FRAÏSSÉ CLASSES WITH SIMPLY CHARACTERIZED BIG RAMSEY STRUCTURES

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Abstract. We formulate a property strengthening the Disjoint Amalgamation Property and prove that every Fraïssé structure in a finite relational language with relation symbols of arity at most two having this property has finite big Ramsey degrees which have a simple characterization. It follows that any such Fraïssé structure admits a big Ramsey structure. Furthermore, we prove indivisibility for every Fraïssé structure in an arbitrary finite relational language satisfying this property. This work offers a streamlined and unifying approach to Ramsey theory on some seemingly disparate classes of Fraïssé structures. Novelties include a new formulation of coding trees in terms of 1-types over initial segments of the Fraïssé structure, and a direct characterization of the degrees without appeal to the standard method of "envelopes".

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Erratum

The Main Theorem and Theorem 3.6 in their full generality are false. We thank Jan Hubička and Matěj Konječný for finding a counterexample showing that the free amalgamation class with two or more 3-ary relations each of which omits a tetrahedron does not have a Fraïssé limit with simply characterized big Ramsey structures. Their example led us to find the error in this paper, which occurs in Theorem 5.17 for relations of arity greater than two.

The Main Theorem and Theorem 3.6 still hold for any Fraïssé class in a finite relational language with relation symbols of arity at most two. Corollary 5.18 holds as written: Fraïssé structures with finitely many relations of any arity satisfying SDAP^+ are indivisible. (Corollary 5.18 is a special case of Theorem 5.12.)

The abstract has been revised to reflect these corrections. The paper itself is currently being revised; no changes have been made in the pages that follow, pending posting of a corrected version.

1. Introduction

In recent years, the Ramsey theory of infinite structures has seen quite an expansion. This area seeks to understand which infinite structures satisfy some analogue of the infinite Ramsey theorem for the natural numbers.

Theorem 1.1 (Ramsey, [40])**.** *Given integers $k, r \geq 1$ and a coloring of the k-element subsets of the natural numbers into r colors, there is an infinite set of natural numbers, N, such that all k-element subsets of N have the same color.*

For infinite structures, exact analogues of Ramsey's theorem usually fail, even when the class of finite substructures has the Ramsey property. This is due to some unseen structure which persists in every infinite substructure isomorphic to the original, but which dissolves when considering Ramsey properties of classes of finite substructures. This was first seen in Sierpiński's use of a well-ordering on the rationals to construct a coloring of unordered pairs of rationals with two colors such that both colors persist in any subcopy of the rationals. The interplay between the well-ordering and the rational order forms additional structure which is in some sense essential, as it persists upon taking any subset forming another dense linear order without endpoints. The quest to characterize and quantify the often hidden but essential structure for infinite structures, more generally, is the area of *big Ramsey degrees.*

Given an infinite structure $\mathbf{M}$, we say that $\mathbf{M}$ has *finite big Ramsey degrees* if for each finite substructure $\mathbf{A}$ of $\mathbf{M}$, there is an integer T such that the following holds: For any coloring of the copies of $\mathbf{A}$ in $\mathbf{M}$ into finitely many colors, there is a substructure $\mathbf{M}'$ of $\mathbf{M}$ such that $\mathbf{M}'$ is isomorphic to $\mathbf{M}$, and the copies of $\mathbf{A}$ in $\mathbf{M}'$ take no more than T colors. When a T having this property exists, the least such value is called the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{M}$, denoted $T(\mathbf{A}, \mathbf{M})$, In particular, if the big Ramsey degree of $\mathbf{A}$ in $\mathbf{M}$ is one, then any finite coloring of the copies of $\mathbf{A}$ in $\mathbf{M}$ is constant on some subcopy of $\mathbf{M}$.

While the area of big Ramsey degrees on infinite structures traces back to Sierpiński's result that the big Ramsey degree for unordered pairs of rationals is at least two, and progress on the rationals and other binary relational structures was made in the decades since, the question of which infinite structures have finite big Ramsey degrees attracted extended interest due to the flurry of results in [29], [30], [38], and [42] in tandem with the publication of [25], in which Kechris, Pestov, and Todorcevic asked for an analogue of their correspondence between the Ramsey property of Fraïssé classes and extreme amenability to the setting of big Ramsey degrees for Fraïssé limits. This was addressed by Zucker in [46], where he proved a connection between Fraïssé limits with finite big Ramsey degrees and completion flows in topological dynamics. Zucker's results apply to *big Ramsey structures*, finite expansions of Fraïssé limits in which the big Ramsey degrees of the Fraïssé limits can be exactly characterized using the additional structure induced by the expanded language. This additional structure involves a well-ordering, and characterizes the essential structure which persists in every infinite subcopy of the Fraïssé limit. It is this essential structure we seek to understand in the study of big Ramsey degrees.

In this paper, we describe an amalgamation property, called the Substructure Disjoint Amalgamation Property (SDAP), forming a strengthened version of disjoint amalgamation. We then characterize the exact big Ramsey degrees for all

Fraïssé limits whose ages have SDAP and which have two additional properties, which we call the Diagonal Coding Tree Property and the Extension Property. Our characterization, together with results of Zucker in [46], imply that Fraïssé limits having these three properties admit big Ramsey structures, and their automorphism groups have metrizable universal completion flows.

Main Theorem. *Let $\mathcal{K}$ be a Fraïssé class in a finite relational language such that $\mathcal{K}$ satisfies SDAP, and such that the Fraïssé limit* $\mathrm{Flim}(\mathcal{K})$ *of $\mathcal{K}$ has the Diagonal Coding Tree Property and the Extension Property. Then* $\mathrm{Flim}(\mathcal{K})$ *has finite big Ramsey degrees and, moreover, admits a big Ramsey structure. Hence, the topological group* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *has a metrizable universal completion flow, which is unique up to isomorphism.*

We say that a Fraïssé structure $\mathbf{K}$ satisfies the Substructure Disjoint Amalgamation Property$^+$ (SDAP$^+$) whenever the age of $\mathbf{K}$ has SDAP, and $\mathbf{K}$ has the Diagonal Coding Tree Property and the Extension Property.

For free amalgamation classes, we provide a stronger property than SDAP called the Substructure Free Amalgamation Property (SFAP). We show that if a Fraïssé class $\mathcal{K}$ has SFAP then its Fraïssé limit has SDAP$^+$, and further, the Fraïssé class $\mathcal{K}^<$ consisting of all ordered expansions of members of $\mathcal{K}$ also has a Fraïssé limit with SDAP$^+$. Thus, we obtain the following corollary.

Corollary 1.2. *Let $\mathcal{K}$ be a Fraïssé class in a finite relational language satisfying SFAP. Then the Fraïssé limit of $\mathcal{K}$,* $\mathrm{Flim}(\mathcal{K})$*, and the Fraïssé limit of $\mathcal{K}^<$,* $\mathrm{Flim}(\mathcal{K}^<)$*, both have finite big Ramsey degrees and, moreover, admit big Ramsey structures. Hence, each of the topological groups* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ *and* $\mathrm{Aut}(\mathrm{Flim}(\mathcal{K}^<))$ *has a metrizable universal completion flow which is unique up to isomorphism.*

An immediate consequence of our main results is that a Fraïssé limit $\mathbf{K}$ with SDAP$^+$ is *indivisible*, by which we mean that every one-element substructure of $\mathbf{K}$ has big Ramsey degree equal to one. In the case when $\mathbf{K}$ has exactly one substructure of size one (up to isomorphism), as happens for instance when the language of $\mathbf{K}$ has no unary relation symbols and there are no "loops" in $\mathbf{K}$, this definition reduces to the usual one for indivisibility of structures like the Rado graph and the Henson graphs (see [42], [27], and [16]). Further, we are able to show that the age of any Fraïssé structure satisfying SDAP$^+$ has ordered expansion with the Ramsey property.

The Main Theorem follows from Theorems 5.22, 6.7, and 6.9 of this paper. Upper bounds for big Ramsey degrees of Fraïssé limits with SDAP$^+$ are found in Theorem 5.22. These bounds are then proved to be exact in Theorem 6.7, where the big Ramsey degrees are characterized by so-called similarity types of diagonal antichains. This characterization is presented at the end of this introduction.

Theorem 6.9 shows that the characterization in Theorem 6.7 implies that the conditions of a theorem of Zucker in [46], guaranteeing existence of big Ramsey structures, hold for any Fraïssé limit satisfying SDAP$^+$. Moreover, the big Ramsey structures can be obtained by simply adding two new relation symbols to the language, similarly to the construction of big Ramsey structures for the rationals and the Rado graph in [46]. Thus, in Theorem 6.10, we obtain a simple characterization of big Ramsey structures for Fraïssé classes with SDAP$^+$ in Theorem.

The Main Theorem provides new classes of examples of big Ramsey structures, recovers results in [9], [29], [30], and extends results in [2] and some of the results in [22] and in [34]. We now discuss several theorems which follow from the Main Theorem, as well as examples of new big Ramsey structures obtained from our results. A fuller description is provided in Section 3.

We show that SFAP holds for free amalgamation classes which are "unrestricted", in a sense that we define in the paper, as well as for those which forbid 3-irreducible substructures, namely, substructures in which any three distinct elements appear in a tuple of which some relation holds. Hence, by Corollary 1.2, these classes as well as their order expansions admit big Ramsey structures. Theorem 3.6 presents this special case of Corollary 1.2.

Particular instances of Theorem 3.6 include classes of structures with binary relations such as graphs, n-partite graphs, and ordered graphs, as well as classes of structures with higher arity relations such as k-uniform hypergraphs. Our examples encompass those unconstrained binary relational structures considered in [30] that have free amalgamation. For structures with relations of arity greater than two, our work recovers results of Balko, Chodounský, Hubička, Konečný and Vena in [2] proving that the generic 3-uniform hypergraph has finite big Ramsey degrees. Furthermore, our results apply to ordered versions of these structures.

Certain Fraïssé structures derived from the rational linear order satisfy SDAP$^+$ and hence, by the Main Theorem, admit big Ramsey structures. Theorem 3.10 shows in particular that $\mathbb{Q}_n$, the rational linear order with a partition into n dense pieces, admits a big Ramsey structure. Theorem 3.10 also shows that the structure $\mathbb{Q}_\mathbb{Q}$ admits a big Ramsey structure, answering a question raised by Zucker at the 2018 Banff Workshop on *Unifying Themes in Ramsey Theory.* This is the dense linear order without endpoints with an equivalence relation such that all equivalence classes are convex copies of the rationals. More generally, Theorem 3.10 applies to members of a natural hierarchy of finitely many convexly ordered equivalence relations, where each successive equivalence relation coarsens the previous one; these also admit big Ramsey structures. Fraïssé structures with finitely many independent linear orders satisfy a slightly weaker property than SDAP$^+$, which we call the *Bounded SDAP$^+$* and discuss in Section 8. We expect that the methods in this paper can be adjusted to find canonical partitions for these structures, but the characterization will be more complex.

Another line of Fraïssé structures derived from the rational linear order are its five reducts. These are the rationals treated as a pure set, the rational linear order itself, the countable structure with the ternary betweenness relation, the countable structure with the ternary circular order, and and that with the quaternary separation relation. We state in Theorem 3.12 that these structures admit big Ramsey structures. We show that the last four satisfy SDAP$^+$; the natural numbers trivially admit big Ramsey structures, by Ramsey's Theorem.

We call structures handled by Theorems 3.10 and 3.12 $\mathbb{Q}$-*like*, as they have enough rigidity, similarly to $\mathbb{Q}$, for SDAP$^+$ to hold and hence, for the proof methods in this paper to apply. Known results in this genre of $\mathbb{Q}$-like structures which our methods recover include Devlin's characterization of the big Ramsey degrees of the rationals [9] as well as results of Laflamme, Nguyen Van Thé, and Sauer in [29] characterizing the big Ramsey degrees of the $\mathbb{Q}_n$. Mašulović recently proved in [34] that the five reducts of the rational linear order have finite big Ramsey degrees, using category theoretic methods to find upper bounds for the degrees. Our results extend his to characterize the exact big Ramsey degrees.

While many of the known big Ramsey degree results use sophisticated versions of Milliken's Ramsey theorem for trees [35], and while proofs using the method of forcing to produce new pigeonhole principles in ZFC have appeared in [10], [11], [13], and [47], there are two novelties to our approach in this paper which produce a clarity about big Ramsey degrees. Given a Fraïssé class $\mathcal{K}$, we fix an enumerated Fraïssé limit of $\mathcal{K}$, which we denote by $\mathbf{K}$. By *enumerated Fraïssé limit*, we mean that the universe of $\mathbf{K}$ is ordered via the natural numbers. The first novelty of our approach is that we work with trees of quantifier-free 1-types (see Definition 4.1) and develop forcing arguments directly on them to prove upper bounds for the big Ramsey degrees. It was suggested to the second author by Sauer during the 2018 BIRS Workshop, *Unifying Themes in Ramsey Theory*, to try moving the forcing methods from [11] and [13] to forcing directly on the structures. Using trees of quantifier-free 1-types seems to come as close as possible to fulfilling this request, as the 1-types allow one to see the essential hidden structure (the interplay of a well-ordering of the universe with first instances where 1-types disagree), whereas working only on the Fraïssé structures, with no reference to 1-types, obscures this central feature of big Ramsey degrees from view. We will be calling such trees *coding trees*, as there will be special nodes, called *coding nodes*, representing the vertices of $\mathbf{K}$: The n-th coding node will be the quantifier-free 1-type of the n-th vertex of $\mathbf{K}$ over the substructure of $\mathbf{K}$ induced on the first $n-1$ vertices of $\mathbf{K}$. (The 0-th coding node is the quantifier-free 1-type of the 0-th vertex over the empty set.) A second novelty of our approach is that we find the exact big Ramsey degrees directly from the trees of 1-types, without appeal to the standard method of "envelopes". This means that the upper bounds which we find via forcing arguments are shown to be exact.

Using trees of quantifier-free 1-types (partially ordered by inclusion) allows us to prove a characterization of big Ramsey degrees for Fraïssé classes with SDAP$^+$ which is a simple extension of the so-called "Devlin types" for the rationals in [9], and of the characterization of the big Ramsey degrees of the Rado graph achieved by Laflamme, Sauer, and Vuksanovic in [30]. Here, we present the characterization for structures without unary relations. The full characterization is given in Theorem 6.7.

Simple Characterization of big Ramsey degrees. *Let $\mathcal{L}$ be a language consisting of finitely many relation symbols, each of arity at least two. Suppose $\mathcal{K}$ is a Fraïssé class in $\mathcal{L}$ such that the Fraïssé limit $\mathbf{K}$ of $\mathcal{K}$ satisfies SDAP$^+$. Fix a structure $\mathbf{A} \in \mathcal{K}$. Let $(\mathbf{A}, <)$ denote $\mathbf{A}$ together with a fixed enumeration $\langle \mathrm{a}_i : i < n \rangle$ of the universe of $\mathbf{A}$. We say that a tree T is a* diagonal tree coding $(\mathbf{A}, <)$ *if the following hold:*

1. *(1) T is a finite tree with n terminal nodes and branching degree two.*
2. *(2) T has at most one branching node in any given level, and no two distinct nodes from among the branching nodes and terminal nodes have the same length. Hence, T has $2n-1$ many levels.*
3. *(3) Let $\langle \mathrm{d}_i : i < n \rangle$ enumerate the terminal nodes in T in order of increasing length. Let $\mathbf{D}$ be the $\mathcal{L}$-structure induced on the set $\{\mathrm{d}_i : i < n\}$ by the increasing bijection from $\langle \mathrm{a}_i : i < n \rangle$ to $\langle \mathrm{d}_i : i < n \rangle$, so that $\mathbf{D} \cong \mathbf{A}$. Let τ_i denote the quantifier-free 1-type of d_i over $\mathbf{D}_i$, the substructure of $\mathbf{D}$ on vertices $\{\mathrm{d}_m : m < i\}$. Given $i < j < k < n$, if d_j and d_k both extend some*

node in T that is at the same level as d_i, then d_j and d_k have the same quantifier-free 1-types over $\mathbf{D}_i$. That is, $\tau_j \restriction \mathbf{D}_i = \tau_k \restriction \mathbf{D}_i$.

Let $\mathcal{D}(\mathbf{A},<)$ denote the number of distinct diagonal trees coding $(\mathbf{A},<)$; let $\mathcal{OA}$ denote a set consisting of one representative from each isomorphism class of ordered copies of $\mathbf{A}$. Then

$$T(\mathbf{A},\mathbf{K}) = \sum_{(\mathbf{A},<)\in\mathcal{OA}} \mathcal{D}(\mathbf{A},<)$$

We see our main contribution as providing a clear and unified analysis of the exact big Ramsey degrees for a wide class of Fraïssé structures with relations of any arity.

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2. Amalgamation properties implying big Ramsey structures

Our Main Theorem shows that for finite relational languages, Fraïssé structures satisfying the Substructure Disjoint Amalgamation Property$^+$ (SDAP$^+$) have simply characterized exact big Ramsey degrees, from which the existence of big Ramsey structures follows. The inspiration for this property comes from a strengthening of the free amalgamation property, which we call the Substructure Free Amalgamation Property (SFAP). We originally found that any Fraïssé structure with an age satisfying SFAP has finite big Ramsey degrees that are characterized in a manner similar to the characterizations, in [30], of big Ramsey degrees for the Rado graph and other unconstrained binary relational structures with disjoint amalgamation. SFAP is satisfied by the ages of all unconstrained relational structures having free amalgamation, as well as by Fraïssé classes with forbidden irreducible and 3-irreducible substructures. The Substructure Disjoint Amalgamation Property (SDAP) is a natural extension of SFAP to a broader collection of Fraïssé classes with disjoint amalgamation. When the Fraïssé limit of an age with SDAP has certain additional properties, which we call the Diagonal Coding Tree Property and the Extension Property (defined in Sections 4 and 5), we say that it has SDAP$^+$. The property SDAP$^+$ ensures a simple characterization of exact big Ramsey degrees and big Ramsey structures.

In Subsection 2.1 we review the basics of Fraïssé theory, the Ramsey property, big Ramsey degrees and big Ramsey structures. More general background on Fraïssé theory can be found in Fraïssé's original paper [19], as well as [20]. The properties SFAP, SDAP and SDAP$^+$ are presented in Subsection 2.2.

2.1. Fraïssé theory, big Ramsey degrees, and big Ramsey structures.

All relations in this paper will be finitary, and all languages will consist of finitely many relation symbols (and no constant or function symbols). We use the set-theoretic notation ω to denote the set of natural numbers, $\{0, 1, 2, \dots\}$ and treat $n \in \omega$ as the set $\{i \in \omega : i < n\}$.

Let $\mathcal{L} = \{R_i : i < I\}$ be a finite language where each R_i is a relation symbol with associated arity $n_i \in \omega$. An *$\mathcal{L}$-structure* is an object

$$\mathbf{M} = \langle \mathrm{M}, R_0^{\mathbf{M}}, \dots, R_{I-1}^{\mathbf{M}} \rangle \tag{1}$$

where M is a nonempty set, called the *universe* of $\mathbf{M}$, and each $R_i^{\mathbf{M}} \subseteq \mathrm{M}^{n_i}$. Finite structures will typically be denoted by $\mathbf{A}, \mathbf{B}$, etc., and their universes by A, B, etc. Infinite structures will typically be denoted by $\mathbf{J}, \mathbf{K}$ and their universes by J, K. We will call the elements of the universe of a structure *vertices*.

An *embedding* between $\mathcal{L}$-structures $\mathbf{M}$ and $\mathbf{N}$ is an injection $\iota : \mathrm{M} \to \mathrm{N}$ such that for each $i < I$ and for all $a_0, \dots, a_{n_i-1} \in \mathrm{M}$,

$$R_i^{\mathbf{M}}(a_0, \dots, a_{n_i-1}) \iff R_i^{\mathbf{N}}(\iota(a_0), \dots, \iota(a_{n_i-1})). \tag{2}$$

A surjective embedding is an *isomorphism*, and an isomorphism from $\mathbf{M}$ to itself $\mathbf{M}$ is an *automorphism*. The set of embeddings of $\mathbf{M}$ into $\mathbf{N}$ is denoted $\mathrm{Emb}(\mathbf{M}, \mathbf{N})$, and the set of automorphisms of $\mathbf{M}$ is denoted $\mathrm{Aut}(\mathbf{M})$. When $\mathrm{M} \subseteq \mathrm{N}$ and the inclusion map is an embedding, we say $\mathbf{M}$ is a *substructure* of $\mathbf{N}$. When there exists an embedding ι from $\mathbf{M}$ to $\mathbf{N}$, the substructure of $\mathbf{N}$ having universe $\iota[\mathrm{M}]$ is called a *copy* of $\mathbf{M}$ in $\mathbf{N}$, and it is a *subcopy* of $\mathbf{N}$ if $\mathbf{M}$ is isomorphic to $\mathbf{N}$. The *age* of $\mathbf{M}$, written $\mathrm{Age}(\mathbf{M})$, is the class of all finite $\mathcal{L}$-structures that embed into $\mathbf{M}$. We write $\mathbf{M} \le \mathbf{N}$ when there is an embedding of $\mathbf{M}$ into $\mathbf{N}$, and $\mathbf{M} \cong \mathbf{N}$ when there is an isomorphism from $\mathbf{M}$ to $\mathbf{N}$.

A class $\mathcal{K}$ of finite structures in a finite relational language is called a *Fraïssé class* if it is nonempty, closed under isomorphisms, hereditary, and satisfies the joint embedding and amalgamation properties. The class $\mathcal{K}$ is *hereditary* if whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \le \mathbf{B}$, then also $\mathbf{A} \in \mathcal{K}$. The class $\mathcal{K}$ satisfies the *joint embedding property* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \le \mathbf{C}$ and $\mathbf{B} \le \mathbf{C}$. The class $\mathcal{K}$ satisfies the *amalgamation property* if for any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and there are embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$. Note that in a finite relational language, there are only countably many finite structures up to isomorphism.

An $\mathcal{L}$-structure $\mathbf{K}$ is called *ultrahomogeneous* if every isomorphism between finite substructures of $\mathbf{K}$ can be extended to an automorphism of $\mathbf{K}$. We call a countably infinite, ultrahomogeneous structure a *Fraïssé structure*. Fraïssé showed [19] that the age of a Fraïssé structure is a Fraïssé class, and that conversely, given a Fraïssé class $\mathcal{K}$, there is, up to isomorphism, a unique Fraïssé structure whose age is $\mathcal{K}$. Such a Fraïssé structure is called the *Fraïssé limit* of $\mathcal{K}$ or the *generic* structure for $\mathcal{K}$.

Throughout this paper, $\mathbf{K}$ will denote the Fraïssé limit of a Fraïssé class $\mathcal{K}$. We will sometimes write $\mathrm{Flim}(\mathcal{K})$ for $\mathbf{K}$. We will assume that $\mathbf{K}$ has universe ω, and call such a structure an *enumerated Fraïssé structure*. For $m < \omega$, we let $\mathbf{K}_m$ denote the substructure of $\mathbf{K}$ with universe $m = \{0, 1, \dots, m-1\}$.

The following amalgamation property will be assumed in this paper: A Fraïssé class $\mathcal{K}$ satisfies the *Disjoint Amalgamation Property* if, given embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is an amalgam $\mathbf{D} \in \mathcal{K}$ with embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$ and moreover, $r[\mathrm{B}] \cap s[\mathrm{C}] = r \circ f[\mathrm{A}] = s \circ g[\mathrm{A}]$. The disjoint amalgamation property is also called the *strong amalgamation property*. It is equivalent to the *strong embedding property*, which requires that for any $\mathbf{A} \in \mathcal{K}$, $v \in \mathrm{A}$, and embedding $\varphi : (\mathbf{A} - v) \to \mathbf{K}$, there are infinitely many different extensions of φ to embeddings of $\mathbf{A}$ into $\mathbf{K}$. (See [7].)

A Fraïssé class has the *Free Amalgamation Property* if it satisfies the Disjoint Amalgamation Property and moreover, the amalgam $\mathbf{D}$ can be chosen so that no tuple satisfying a relation in $\mathbf{D}$ includes elements of both $r[\mathrm{B}] \setminus r \circ f[\mathrm{A}]$ and $s[\mathrm{C}] \setminus s \circ g[\mathrm{A}]$; in other words, $\mathbf{D}$ has no additional relations on its universe other than those inherited from $\mathbf{B}$ and $\mathbf{C}$.

For languages $\mathcal{L}_0$ and $\mathcal{L}_1$ such that $\mathcal{L}_0 \cap \mathcal{L}_1 = \emptyset$, and given Fraïssé classes $\mathcal{K}_0$ and $\mathcal{K}_1$ in $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively, the *free superposition* of $\mathcal{K}_0$ and $\mathcal{K}_1$ is the Fraïssé class consisting of finite $(\mathcal{L}_0 \cup \mathcal{L}_1)$-structures $\mathbf{A}$ such that the $\mathcal{L}_i$-reduct of $\mathbf{A}$ is in $\mathcal{K}_i$, for each $i < 2$. (See also [6] and [22].) Note that the free superposition of $\mathcal{K}_0$ and $\mathcal{K}_1$ has free amalgamation if and only if each $\mathcal{K}_i$ has free amalgamation; and similarly for disjoint amalgamation.

Given a Fraïssé class $\mathcal{K}$ and substructures $\mathbf{M}, \mathbf{N}$ of $\mathbf{K}$ (finite or infinite) with $\mathbf{M} \leq \mathbf{N}$, we use $\binom{\mathbf{N}}{\mathbf{M}}$ to denote the set of all substructures of $\mathbf{N}$ which are isomorphic to $\mathbf{M}$. Given $\mathbf{M} \leq \mathbf{N} \leq \mathbf{O}$, substructures of $\mathbf{K}$, we write

$$\mathbf{O} \to (\mathbf{N})^{\mathbf{M}}_{\ell}$$

to denote that for each coloring of $\binom{\mathbf{O}}{\mathbf{M}}$ into ℓ colors, there is an $\mathbf{N}' \in \binom{\mathbf{O}}{\mathbf{N}}$ such that $\binom{\mathbf{N}'}{\mathbf{M}}$ is *monochromatic*, meaning that all members of $\binom{\mathbf{N}'}{\mathbf{M}}$ have the same color.

Definition 2.1. A Fraïssé class $\mathcal{K}$ has the *Ramsey property* if for any two structures $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and any $\ell \geq 2$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{B} \leq \mathbf{C}$ such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{\ell}$.

Equivalently, $\mathcal{K}$ has the Ramsey property if for any two structures $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$,

$$\forall \ell \geq 2, \quad \mathbf{K} \to (\mathbf{B})^{\mathbf{A}}_{\ell}. \tag{3}$$

This equivalent formulation makes comparison with big Ramsey degrees, below, quite clear.

Definition 2.2 ([25])**.** Given a Fraïssé class $\mathcal{K}$ and its Fraïssé limit $\mathbf{K}$, for any $\mathbf{A} \in \mathcal{K}$, write

$$\forall \ell \geq 1, \quad \mathbf{K} \to (\mathbf{K})^{\mathbf{A}}_{\ell,T} \tag{4}$$

when there is an integer $T \geq 1$ such that for any integer $\ell \geq 1$, given any coloring of $\binom{\mathbf{K}}{\mathbf{A}}$ into ℓ colors, there is a substructure $\mathbf{K}'$ of $\mathbf{K}$, isomorphic to $\mathbf{K}$, such that $\binom{\mathbf{K}'}{\mathbf{A}}$ takes no more than T colors. We say that $\mathbf{K}$ has *finite big Ramsey degrees* if for each $\mathbf{A} \in \mathcal{K}$, there is an integer $T \geq 1$ such that equation (4) holds. For a given finite $\mathbf{A} \leq \mathbf{K}$, when such a T exists, we let $T(\mathbf{A}, \mathbf{K})$ denote the least one, and call this number the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{K}$.

Comparing equations (3) and (4), we see that the difference between the Ramsey property and having finite big Ramsey degrees is that the former finds a substructure of $\mathbf{K}$ isomorphic to the *finite* structure $\mathbf{B}$ in which all copies of $\mathbf{A}$ have the *same* color, while the latter finds an *infinite* substructure of $\mathbf{K}$ which is isomorphic to $\mathbf{K}$ in which the copies of $\mathbf{A}$ take *few* colors. It is only when $T(\mathbf{A}, \mathbf{K}) = 1$ that there is a subcopy of $\mathbf{K}$ in which all copies of $\mathbf{A}$ have the same color.

It is normally the case that for structures $\mathbf{A}$ with universe of size greater than one, $T(\mathbf{A}, \mathbf{K})$ is at least two, if it exists at all. The fundamental reason for this stems from Sierpińksi's example that $T(2, \mathbb{Q}) \geq 2$: The enumeration of the universe

ω of **K** plays against the relations in the structure to preserve more than one color in every subcopy of **K**.

On the other hand, many classes $\mathcal{K}$ are known to have singleton structures (that is, structures with universe consisting of one element) with big Ramsey degree one; when this holds for every (isomorphism type of) singleton structure in $\mathcal{K}$, we say that **K** is *indivisible*.

Definition 2.3. A Fraïssé structure **K** is *indivisible* if for every singleton substructure **A** of **K**, $T(\mathbf{A}, \mathbf{K})$ exists and equals one.

Note that when there is only one quantifier-free 1-type over the empty set satisfied by elements of **K**, so that **K** has exactly one singleton substructure up to isomorphism, indivisibility amounts to saying that for any partition of the universe of **K** into finitely many pieces, there is a subcopy of **K** contained in one of the pieces. Indivisibility has been proved for many structures, including the triangle-free Henson graph in [27], the k-clique-free Henson graphs for all $k \geq 4$ in [16], more general binary relational structures in [41], and for k-uniform hypergraphs, $k \geq 3$, that omit finite substructures in which all unordered triples of vertices are contained in at least one k-edge in [17]. For a much broader discussion of Fraïssé structures and indivisibility, the reader is referred to Nguyen Van Thé's Habilitation [39].

A proof that **K** has finite big Ramsey degrees amounts to showing that the numbers $T(\mathbf{A}, \mathbf{K})$ exist by finding upper bounds for them. When a method for producing the numbers $T(\mathbf{A}, \mathbf{K})$ is given, we will say that the *exact big Ramsey degrees* have been *characterized*. In all known cases where exact big Ramsey degrees have been characterized, this has been done by finding *canonical partitions* for the finite substructures of **K**.

Definition 2.4 (Canonical Partition)**.** Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit **K**, and let $\mathbf{A} \in \mathcal{K}$ be given. A partition $\{P_i : i < n\}$ of $\binom{\mathbf{K}}{\mathbf{A}}$ is a *canonical partition* if the following hold:

(1) For every subcopy **J** of **K** and each $i < n$, $P_i \cap \binom{\mathbf{J}}{\mathbf{A}}$ is non-empty. This property is called *persistence*.
(2) For each finite coloring γ of $\binom{\mathbf{K}}{\mathbf{A}}$ there is a subcopy $\mathbf{J}_\gamma$ of **K** such that for each $i < n$, all members of $P_i \cap \binom{\mathbf{J}_\gamma}{\mathbf{A}}$ are assigned the same color by γ.

Remark 2.5. In many papers on big Ramsey degrees, including the foundational results in [9], [42], and [30], authors color *copies* of a given $\mathbf{A} \in \mathcal{K}$ inside **K**, working with Definition 2.2. In some papers, especially those with very direct ties to topological dynamics of automorphism groups as in [46] and [47], the authors color *embeddings* of **A** into **K**. The relationship between these approaches is simple: A structure $\mathbf{A} \in \mathcal{K}$ has big Ramsey degree t for copies if and only if **A** has big Ramsey degree $t \cdot |\mathrm{Aut}(\mathbf{A})|$ for embeddings. Thus, one can use whichever formulation most suits the context. Furthermore, we show in Theorem 6.9 that there is a simple way of recovering Zucker's criterion for existence of big Ramsey structures (which uses colorings of embeddings; see Theorem 7.1 in [46]) from canonical partitions (via trees of 1-types) for colorings of copies of a structure.

The majority of results on big Ramsey degrees have been proved using some auxiliary structure, usually trees, and recently sequences of parameter words (see [22]), to characterize the persistent superstructures which code the finite structure **A**. The exception is the recent use of category-theoretic approaches (see for instance

[5], [33], and [34]). These superstructures fade away in the case of finite structures with the Ramsey property. An example of how this works can be seen in Theorem 7.2, where we recover the ordered Ramsey property for ages of Fraïssé structures with SDAP$^+$ from their big Ramsey degrees. However, for big Ramsey degrees of Fraïssé limits, these superstructures possess some essential features which persist, leading to big Ramsey degrees greater than one. The following notion of Zucker deals with such superstructures via expanded languages.

Let $\mathcal{L}$ be a relational language, M a set, $\mathbf{N}$ an $\mathcal{L}$-structure, and $\iota : \mathrm{M} \to \mathrm{N}$ an injection. Write $\mathbf{N} \cdot \iota$ for the unique $\mathcal{L}$-structure having underlying set M such that ι is an embedding of $\mathbf{N} \cdot \iota$ into $\mathbf{N}$.

Definition 2.6 (Zucker, [46])**.** Let $\mathbf{K}$ be a Fraïssé structure in a relational language $\mathcal{L}$ with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$. We say that $\mathbf{K}$ *admits a big Ramsey structure* if there is a relational language $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-structure $\mathbf{K}^*$ so that the following hold:

(1) The reduct of $\mathbf{K}^*$ to the language $\mathcal{L}$ equals $\mathbf{K}$.
(2) Each $\mathbf{A} \in \mathcal{K}$ has finitely many expansions to an $\mathcal{L}^*$-structure $\mathbf{A}^* \in \mathrm{Age}(\mathbf{K}^*)$; denote the set of such expansions by $\mathbf{K}^*(\mathbf{A})$.
(3) For each $\mathbf{A} \in \mathcal{K}$, $T(\mathbf{A}, \mathbf{K}) \cdot |\mathrm{Aut}(\mathbf{A})| = |\mathbf{K}^*(\mathbf{A})|$
(4) For each $\mathbf{A} \in \mathcal{K}$, the function $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to \mathbf{K}^*(\mathbf{A})$ given by $\gamma(\iota) = \mathbf{K}^* \cdot \iota$ witnesses the fact that
$$T(\mathbf{A}, \mathbf{K}) \cdot |\mathrm{Aut}(\mathbf{A})| \geq |\mathbf{K}^*(\mathbf{A})|,$$
in the following sense: For every subcopy $\mathbf{K}'$ of $\mathbf{K}$, the image of the restriction of γ to $\mathrm{Emb}(\mathbf{A}, \mathbf{K}')$ has size $|\mathbf{K}^*(\mathbf{A})|$.

Such a structure $\mathbf{K}^*$ is called a *big Ramsey structure* for $\mathbf{K}$.

Note that the definition of a big Ramsey structure for $\mathbf{K}$ presupposes that $\mathbf{K}$ has finite big Ramsey degrees. The big Ramsey structure $\mathbf{K}^*$, when it exists, is a device for storing information about all the big Ramsey degrees in $\mathbf{K}$ together in a uniform way.

While the study of big Ramsey degrees has been progressing for many decades, a recent compelling motivation for finding big Ramsey structures is the following theorem.

Theorem 2.7 (Zucker, [46])**.** *Let* $\mathbf{K}$ *be a Fraïssé structure which admits a big Ramsey structure, and let* $G = \mathrm{Aut}(\mathbf{K})$*. Then the topological group* G *has a metrizable universal completion flow, which is unique up to isomorphism.*

This theorem answered a question in [25] which asked for an analogue, in the context of finite big Ramsey degrees, of the Kechris-Pestov-Todorcevic correspondence between the Ramsey property for a Fraïssé class and extreme amenability of the automorphism group of its Fraïssé limit; Zucker's theorem provides a similar connection between finite big Ramsey degrees and universal completion flows. The notion of big Ramsey degree in [46] involves colorings of embeddings of structures instead of just colorings of substructures. As described in Remark 2.5, this poses no problem when applying our results on big Ramsey degrees, which involve coloring copies of a structure, to Theorem 2.7.

2.2. **The Substructure Disjoint Amalgamation Property$^+$.** We now present the amalgamation properties that lead to simple characterizations of exact big Ramsey degrees in this paper. Recall that given a Fraïssé class $\mathcal{K}$ in a finite relational language $\mathcal{L}$, we let $\mathbf{K}$ denote an enumerated Fraïssé limit of $\mathcal{K}$ with underlying set ω. All results will hold regardless of which enumeration is chosen. We make the following conventions and assumptions, which will hold in the rest of this paper.

All types will be quantifier-free 1-types, over a finite parameter set, that are realizable in $\mathbf{K}$. With one exception, all such types will be complete; the exception is the case of "passing types", defined in Section 4, which may be partial. Complete types will be denoted simply "tp".

We will assume that for any relation symbol R in $\mathcal{L}$, $R^{\mathbf{K}}(\bar{a})$ can hold only for tuples $\bar{a}$ of *distinct* elements of ω. In particular, we assume our structures have no loops. We further assume that all relations in $\mathbf{K}$ are *non-trivial*: This means that for each relation symbol R in $\mathcal{L}$, there exists a k-tuple $\bar{a}$ of (distinct) elements of ω such that $R^{\mathbf{K}}(\bar{a})$ holds, and a tuple $\bar{b}$ of (distinct) elements of ω such that $\neg R^{\mathbf{K}}(\bar{b})$ holds. Since $\mathcal{K}$ has disjoint amalgamation by assumption, non-triviality will imply that there are infinitely many k-tuples from ω that satisfy $R^{\mathbf{K}}$, and infinitely many that do not. We will further hold to the convention that if $\mathcal{L}$ has any unary relation symbols, then letting $R_0, \dots, R_{n-1}$ list them, we have that $n \geq 2$ and for each $\mathbf{A} \in \mathcal{K}$, for each $a \in \mathrm{A}$, $R_i^{\mathbf{A}}(a)$ holds for exactly one $i < n$. By possibly adding new unary relation symbols to the language, any Fraïssé class with unary relations can be assumed to meet this convention. Finally, we assume that there is at least one non-unary relation symbol in $\mathcal{L}$. This poses no real restriction, as whenever a finite language has only unary relation symbols, any disjoint amalgamation class in that language will have a Fraïssé limit that consists of finitely many disjoint copies of ω, with vertices in a given copy all realizing the same quantifier-free 1-type over the empty set. In this case, finitely many applications of Ramsey's Theorem will prove the existence of finite big Ramsey degrees.

We now present the Substructure Free Amalgamation Property, a strengthening of free amalgamation for a Fraïssé class $\mathcal{K}$, whose presence implies that the Fraïssé limit $\mathbf{K}$ admits a big Ramsey structure. This property also provides the intuition behind the more general amalgamation property SDAP (Definition 2.10), laying the foundation for the main ideas of this paper.

Definition 2.8 (SFAP)**.** A Fraïssé class $\mathcal{K}$ has the *Substructure Free Amalgamation Property (SFAP)* if $\mathcal{K}$ has free amalgamation, and given $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, the following holds: Suppose

(1) $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$;
(2) $\mathbf{A}$ is a substructure of $\mathbf{B}$ and σ and τ are 1-types over $\mathbf{B}$ with $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$; and
(3) $\mathbf{B}$ is a substructure of $\mathbf{D}$ which extends $\mathbf{B}$ by one vertex, say v', such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$, and $\mathbf{E}$ adds no other relations over D.

The definition of SFAP can be stated using embeddings rather than substructures in the standard way. We remark that requiring $\mathbf{C}$ in (1) to have only two more vertices than $\mathbf{A}$ is sufficient for all our uses of the property in proofs of big Ramsey degrees, and hence we have not formulated the property for $\mathbf{C}$ of arbitrary finite size.

Remark 2.9. SFAP is equivalent to free amalgamation along with a model-theoretic property that may be termed *free 3-amalgamation*, a special case of the *disjoint 3-amalgamation* property defined in [28]: In the definition of disjoint n-amalgamation in Section 3 of [28], take $n = 3$ and impose the further condition that the "solution" or 3-amalgam disallows any relations (in any realization of the solution) that were not already stipulated in the initial 3-amalgamation "problem". Kruckman shows in [28] that if the age of a Fraïssé limit $\mathbf{K}$ has disjoint amalgamation and disjoint 3-amalgamation, then $\mathbf{K}$ exhibits a model-theoretic tameness property called *simplicity*. As free amalgamation is a special kind of disjoint amalgamation, and free 3-amalgamation a special kind of disjoint 3-amalgamation, our work provides a large class of examples of simple Fraïssé structures having finite big Ramsey degrees.

SFAP ensures that a finite substructure of a given enumerated Fraïssé structure can be extended as desired without any requirements on its configuration inside the larger structure. SFAP precludes any need for the so-called "witnessing properties" which were necessary for the proofs of finite big Ramsey degrees for constrained binary free amalgamation classes, as in the k-clique-free Henson graphs in [13] and [11], and the recent more general extensions in [47]. Free amalgamation classes with forbidden 3-irreducible substructures satisfy SFAP, as shown in Proposition 3.2.

The next amalgamation property extends SFAP to disjoint amalgamation classes. In the definition, we again use substructures rather than embeddings.

Definition 2.10 (SDAP)**.** A Fraïssé class $\mathcal{K}$ has the *Substructure Disjoint Amalgamation Property (SDAP)* if $\mathcal{K}$ has disjoint amalgamation, and the following holds: Given $\mathbf{A}, \mathbf{C} \in \mathcal{K}$, suppose that $\mathbf{A}$ is a substructure of $\mathbf{C}$, where $\mathbf{C}$ extends $\mathbf{A}$ by two vertices, say v and w. Then there exist $\mathbf{A}', \mathbf{C}' \in \mathcal{K}$, where $\mathbf{A}'$ contains a copy of $\mathbf{A}$ as a substructure and $\mathbf{C}'$ is a disjoint amalgamation of $\mathbf{A}'$ and $\mathbf{C}$ over $\mathbf{A}$, such that letting v', w' denote the two vertices in $\mathrm{C}' \setminus \mathrm{A}'$ and assuming (1) and (2), the conclusion holds:

(1) Suppose $\mathbf{B} \in \mathcal{K}$ is any structure containing $\mathbf{A}'$ as a substructure, and let σ and τ be 1-types over $\mathbf{B}$ satisfying $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v'/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w'/\mathbf{A}')$,
(2) Suppose $\mathbf{D} \in \mathcal{K}$ extends $\mathbf{B}$ by one vertex, say v'', such that $\mathrm{tp}(v''/\mathbf{B}) = \sigma$.

Then there is an $\mathbf{E} \in \mathcal{K}$ extending $\mathbf{D}$ by one vertex, say w'', such that $\mathrm{tp}(w''/\mathbf{B}) = \tau$ and $\mathbf{E} \restriction (\mathrm{A} \cup \{v'', w''\}) \cong \mathbf{C}$.

Remark 2.11. We note that SFAP implies SDAP, taking $\mathbf{A}' = \mathbf{A}$ and $\mathbf{C}' = \mathbf{C}$ and because disjoint amalgamation is implied by free amalgamation. Further, it follows from their definitions that SFAP and SDAP are each preserved under free superposition.

Example 2.12. The idea behind allowing for an extension $\mathbf{A}'$ of $\mathbf{A}$ in the definition of SDAP is most simply demonstrated for the Fraïssé class $\mathcal{LO}$ of finite linear orders. Given $\mathbf{A}, \mathbf{C} \in \mathcal{LO}$, suppose v, w are the two vertices of $\mathrm{C} \setminus \mathrm{A}$ and suppose that $v < w$ holds in $\mathbf{C}$. We can require $\mathbf{A}'$ to be some extension of $\mathbf{A}$ in $\mathcal{K}$ containing some vertex u so that the formula $(x < u)$ is in $\mathrm{tp}(v/\mathbf{A}')$ and $(u < x)$ is in $\mathrm{tp}(w/\mathbf{A}')$, where x is a variable. Then given any 1-types σ, τ extending $\mathrm{tp}(v/\mathbf{A}'), \mathrm{tp}(w/\mathbf{A}')$, respectively, over some structure $\mathbf{B}$ containing $\mathbf{A}'$ as a substructure, any two vertices v', w' satisfying σ, τ will automatically satisfy $v' < w'$, thus producing a copy of $\mathbf{C}$ extending $\mathbf{A}$.

In the case of finitely many independent linear orders, we can similarly produce an $\mathbf{A}'$ which ensures that *any* vertices v', w' satisfying such σ, τ as above produce a copy of $\mathbf{C}$ extending $\mathbf{A}$. In more general cases, the use of $\mathbf{A}'$ only ensures that *there exist* such vertices v', w'.

Remark 2.13. Ivanov [24] and independently, Kechris and Rosendal [26], have formulated a weakening of the amalgamation property which is called *almost amalgamation* in [24] and *weak amalgamation* in [26]. This property arises in the context of generic automorphisms of countable structures. In the presence of disjoint amalgamation, SDAP may be thought of as a ternary version of weak amalgamation (one of several possible such versions), and as a "weak" version of the disjoint 3-amalgamation property from [28] (again, one of several possible such weakenings).

Remark 2.14. We note that we could have used the definition of the free 3-amalgamation property from Remark 2.9, and of an appropriately formulated version of a "weak" disjoint 3-amalgamation property as in Remark 2.13, to simplify some of the proofs in the next section, which gives examples of Fraïssé classes with SFAP and SDAP. We have chosen to use Definitions 2.8 and 2.10 instead, as they more closely mirror what is needed in the constructions proving the Main Result.

In Section 4 onwards we will be working with so-called *coding trees* of 1-types, which represent subcopies of a given Fraïssé limit $\mathbf{K}$. For Fraïssé structures in languages with relation symbols of arity greater than two, a priori, these trees may have unbounded branching; this has posed a serious obstacle to the general development of their big Ramsey theory. However, for all classes with SFAP and for all classes with SDAP which we have investigated, one can construct subtrees with bounded branching which still represent $\mathbf{K}$. Accordingly, we formulate the following strengthened version of SDAP, which imposes conditions on the branching in a coding tree for $\mathbf{K}$. The notions regarding coding trees of 1-types require some introduction, so we refer the reader to Sections 4 and 5 for their definitions rather than reproducing everything here. We mention only that diagonal trees are skew trees which have binary splitting (see Definition 4.19). The Diagonal Coding Tree Property is defined in Section 4 (Definition 4.23) and the Extension Property is defined in Section 5 (Definition 5.5).

Definition 2.15 (SDAP$^+$)**.** A Fraïssé structure $\mathbf{K}$ has the *Substructure Disjoint Amalgamation Property*$^+$ *(SDAP$^+$)* if its age $\mathcal{K}$ satisfies SDAP, and $\mathbf{K}$ has the Diagonal Coding Tree Property and the Extension Property.

We note that while the Diagonal Coding Tree Property and Extension Property are defined in terms of an enumerated Fraïssé structure, they are independent of the chosen enumeration, and hence SDAP$^+$ is a property of a Fraïssé structure itself.

We have already seen that SFAP implies SDAP. It will be shown in Theorem 4.28 that SFAP in fact implies SDAP$^+$, recalling our assumption throughout that the language contains a non-unary relation symbol. A coding tree version of SDAP$^+$ is presented in Definition 4.26, and is implied by Definition 2.15. This coding tree version will be used in the proofs in this paper.

The motivation behind SDAP$^+$ was to distill the essence of those Fraïssé classes for which the forcing arguments in Section 5 work. As such, it yields big Ramsey degrees which have simple characterizations, similar to those of the rationals and the Rado graph. It is known that SDAP$^+$, and even SDAP, are not necessary for obtaining finite big Ramsey degrees. For instance, generic k-clique-free graphs [11] and the generic partial order [22] have been shown to have finite big Ramsey degrees, and their ages do not have SDAP. A catalogue of these and other such results will be presented at the end of Section 3. The focus of this paper is on characterizing the big Ramsey degrees of those Fraïssé classes for which simple forcing methods suffice.

3. Examples of Fraïssé classes satisfying SDAP$^+$

We now investigate Fraïssé classes for which our Main Theorem holds. Such classes seem to fall roughly into two categories: Free amalgamation classes of relational structures in which any forbidden substructures are 3-irreducible (Definition 3.1) or which are unrestricted (Definition 3.3), as well as their ordered expansions; and disjoint amalgamation classes which are in some sense "$\mathbb{Q}$-like". It follows from the Main Theorem that all of these Fraïssé classes have Fraïssé limits that admit big Ramsey structures.

In this section, we will be verifying that various collections of Fraïssé classes satisfy SDAP. We will show later that most of these Fraïssé classes in fact have Fraïssé limits satisfying SDAP$^+$, after we have developed the machinery of coding trees of 1-types in Section 4. At the end of this section, we provide a catalogue of Fraïssé structures which have been investigated for big Ramsey degrees. The list is non-exhaustive, as research is ongoing, but it provides a view of many of the main results currently known.

First, we consider free amalgamation classes. The following definition appears in [8], and occurs implicitly in work on indivisibility in [17].

Definition 3.1. Let $r \geq 2$, and let $\mathcal{L}$ be a finite relational language. An $\mathcal{L}$-structure $\mathbf{F}$ is *r-irreducible* if for any r distinct elements $a_0, \dots, a_{r-1}$ in F there is some $R \in \mathcal{L}$ and k-tuple $\bar{p}$ with entries from F, where $k \geq r$ is the arity of R, such that each a_i, $i < r$, is among the entries of $\bar{p}$, and $R^{\mathbf{F}}(\bar{p})$ holds. We say $\mathbf{F}$ is *irreducible* when $\mathbf{F}$ is 2-irreducible.

Note that for $r > \ell \geq 2$, a structure that is r-irreducible need not be ℓ-irreducible. This is because for any structure $\mathbf{F}$ such that $|\mathrm{F}| < r$, it is vacuously the case that $\mathbf{F}$ is r-irreducible, but if $|\mathrm{F}| \geq \ell$, then $\mathbf{F}$ may not be ℓ-irreducible.

Given a set $\mathcal{F}$ of finite $\mathcal{L}$-structures, let $\mathrm{Forb}(\mathcal{F})$ denote the class of finite $\mathcal{L}$-structures $\mathbf{A}$ such that no member of $\mathcal{F}$ embeds into $\mathbf{A}$. It is a standard fact that a Fraïssé class $\mathcal{K}$ is a free amalgamation class if and only if $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ for some set $\mathcal{F}$ of finite irreducible $\mathcal{L}$-structures. (See [44] for a proof). When $\mathcal{F}$ is a set of finite $\mathcal{L}$-structures such that members of $\mathcal{F}$ are both irreducible and 3-irreducible, $\mathrm{Forb}(\mathcal{F})$ furthermore has SFAP.

Proposition 3.2. *Let $\mathcal{L}$ be a finite relational language and $\mathcal{F}$ a (finite or infinite) collection of finite $\mathcal{L}$-structures which are irreducible and 3-irreducible. Then* $\mathrm{Forb}(\mathcal{F})$ *satisfies SFAP. Hence the Fraïssé limit of* $\mathrm{Forb}(\mathcal{F})$ *has SDAP$^+$.*

Proof. Since the structures in $\mathcal{F}$ are irreducible, $\mathrm{Forb}(\mathcal{F})$ is a free amalgamation class.

Fix $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathrm{Forb}(\mathcal{F})$ with $\mathbf{A}$ a substructure of both $\mathbf{B}$ and $\mathbf{C}$ and $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$. Let σ, τ be realizable 1-types over $\mathbf{B}$ with $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$ and $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$. Suppose $\mathbf{D} \in \mathrm{Forb}(\mathcal{F})$ is a 1-vertex extension of $\mathbf{B}$ realizing σ. Thus, $\mathrm{D} = \mathrm{B} \cup \{v'\}$ for some v' such that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$.

Extend $\mathbf{D}$ to an $\mathcal{L}$-structure $\mathbf{E}$ by one vertex w' satisfying $\mathrm{tp}(w'/\mathbf{B}) = \tau$ such that for each relation symbol $R \in \mathcal{L}$, letting k denote the arity of R, we have the following:

(a) For each k-tuple $\bar{p}$ with entries from $\mathrm{A} \cup \{v', w'\}$, let $\bar{q}$ be the k-tuple with entries from $\mathrm{A} \cup \{v, w\}$ such that each occurrence of v', w' in $\bar{p}$ (if any) is replaced by v, w, respectively, and all other entries remain the same. Then we require that $R^{\mathbf{E}}(\bar{p})$ holds if and only if $R^{\mathbf{C}}(\bar{q})$ holds.

(b) If $k \geq 3$, then for each $b \in \mathrm{B} \setminus \mathrm{A}$ and each k-tuple $\bar{p}$ with entries from E such that b, v', w' are among the entries of $\bar{p}$, we require that $\neg R^{\mathbf{E}}(\bar{p})$ holds.

It follows from (a) that $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$. It remains to show that $\mathbf{E}$ is a member of $\mathrm{Forb}(\mathcal{F})$. To do so, it suffices to show that no $\mathbf{F} \in \mathcal{F}$ embeds into $\mathbf{E}$.

Suppose toward a contradiction that some $\mathbf{F} \in \mathcal{F}$ embeds into $\mathbf{E}$. Let $\mathbf{F}'$ denote an embedded copy of $\mathbf{F}$, with universe $\mathrm{F}' \subseteq \mathrm{E}$. For what follows, it helps to recall that $\mathrm{E} = \mathrm{B} \cup \{v', w'\}$. Since $\mathbf{D}$ is in $\mathrm{Forb}(\mathcal{F})$, $\mathbf{F}$ does not embed into $\mathbf{D}$, so F' cannot be contained in D. Hence w' must be in F'. Likewise, since τ is a realizable 1-type over $\mathbf{B}$, the substructure $\mathbf{E} \restriction (\mathrm{B} \cup \{w'\})$ is in $\mathrm{Forb}(\mathcal{F})$ and hence does not contain a copy of $\mathbf{F}$. Therefore, v' must be in F'. By (a), since $\mathbf{C}$ is in $\mathrm{Forb}(\mathcal{F})$, the substructure $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\})$ does not contain a copy of $\mathbf{F}$. Hence there must be some $b \in \mathrm{B} \setminus \mathrm{A}$ such that b is in F'. Since $\mathbf{F}$ is 3-irreducible, there must be some relation symbol $R \in \mathcal{L}$ with arity $k \geq 3$, and some k-tuple $\bar{p}$ with entries from F' and with b, v', w' among its entries, such that $R^{\mathbf{F}'}(\bar{p})$ holds. However, (b) implies $\neg R^{\mathbf{E}}(\bar{b})$ holds, contradicting that $\mathbf{F}'$ is a copy of $\mathbf{F}$ in $\mathbf{E}$. Therefore, $\mathbf{F}$ does not embed into $\mathbf{E}$. It follows that $\mathbf{E}$ is a member of $\mathrm{Forb}(\mathcal{F})$.

We have established that $\mathrm{Forb}(\mathcal{F})$ has SFAP. Theorem 4.28 implies that the Fraïssé limit of $\mathrm{Forb}(\mathcal{F})$ has SDAP^+. □

We now consider a property of a Fraïssé class which says, essentially, that substructures of its Fraïssé limit that have size equal to the arities of the relation symbols in the language are independent of one another.

Definition 3.3. A Fraïssé class $\mathcal{K}$ is *unrestricted* if $\mathcal{K}$ is a free superposition of finitely many Fraïssé classes $\mathcal{K}_i$, each of which has the following property: All relation symbols in the language $\mathcal{L}_i$ of $\mathcal{K}_i$ have some fixed arity $k_i \geq 1$, and $\mathcal{K}_i = \mathrm{Forb}(\mathcal{F}_i)$ for some finite collection $\mathcal{F}_i$ of finite $\mathcal{L}_i$-structures such that each member of $\mathcal{F}_i$ has universe of size equal to k_i.

In [30], Laflamme, Sauer, and Vuksanovic characterized the exact big Ramsey degrees for structures that they call *countable universal homogeneous structures over a binary language*. These are precisely the structures in a finite binary relational language whose ages are unrestricted. Some of the classes considered in [30], such as the class of finite graphs, have free amalgamation, while others, such as the class of finite tournaments, have disjoint but not free amalgamation. We now show that any unrestricted Fraïssé class with free amalgamation satisfies SFAP.

Proposition 3.4. *Let $\mathcal{K}$ be an unrestricted Fraïssé class with free amalgamation. Then $\mathcal{K}$ satisfies SFAP. Hence the Fraïssé limit of $\mathcal{K}$ has SDAP^+.*

Proof. Let $\mathcal{L}$ be a finite relational language, k the arity of each relation symbol in $\mathcal{L}$, and $\mathcal{F}$ a finite collection of $\mathcal{L}$-structures such that for each $\mathbf{F} \in \mathcal{F}$, $|\mathrm{F}| = k$. Suppose $\mathcal{K}$ is a free amalgamation class such that $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$. By Remark 2.11, it suffices to prove the result for such $\mathcal{K}$.

Suppose $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D} \in \mathcal{K}$, v, w, v', and σ, τ satisfy (1)-(3) of Definition 2.8. Let ρ be the 1-type of w over $\mathbf{C} \restriction (\mathrm{A} \cup \{v\})$. Then take $\mathbf{E}$ to be the $\mathcal{L}$-structure extending $\mathbf{D}$ by one vertex, say w', such that $\mathrm{tp}(w'/\mathbf{B}) = \tau$, such that $\mathrm{tp}(w'/(\mathbf{D} \restriction (A \cup \{v'\}))$ is the 1-type obtained by substituting v' for v in ρ, and such that no relations hold of tuples from E except those specified by σ, τ or ρ. Then $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$. It remains to show that $\mathbf{E}$ is in $\mathcal{K}$.

Suppose not. Then there is some $\mathbf{F} \in \mathcal{F}$, with $|\mathrm{F}| = k$, such that $\mathbf{F}$ embeds in $\mathbf{E}$. Let $\mathbf{F}'$ be a copy of $\mathbf{F}$ in $\mathbf{E}$. Since $\mathcal{K}$ has free amalgamation, $\mathbf{F}'$ is irreducible. Hence there must be some $R \in \mathcal{L}$ and $|\mathrm{F}'|$-tuple $\bar{p}$ enumerating the elements of F', possibly with repetition, such that $R^{\mathbf{F}'}(\bar{p})$ holds. By the construction of $\mathbf{E}$, the entries of $\bar{p}$ must come from one of $\mathrm{B} \cup \{v'\}$, $\mathrm{B} \cup \{w'\}$ or C. This means that $\mathbf{F}'$ is contained in the restriction of $\mathbf{E}$ to one of these sets, which is a contradiction. □

Remark 3.5. A similar proof shows that the Fraïssé class of finite tournaments, and more generally any unrestricted binary Fraïssé class, satisfy SDAP. It follows from work in [30] that these classes also satisfy SDAP$^+$.

By Proposition 3.7 below, the Fraïssé class of finite linear orders satisfies SDAP. As SFAP implies SDAP, and SDAP is preserved under free superpositions (see Remark 2.11), the ordered expansion $\mathcal{K}^<$ of any Fraïssé class $\mathcal{K}$ with SFAP will also have SDAP. In Theorem 4.28 and Lemma 4.31 we will show that whenever a Fraïssé class $\mathcal{K}$ has SFAP, the Fraïssé limits of $\mathcal{K}$ and $\mathcal{K}^<$ satisfy SDAP$^+$. Applying Propositions 3.2 and 3.4 and the Main Theorem, we then obtain the following.

Theorem 3.6. *Let $\mathcal{L}$ be a finite relational language, $\mathcal{K}$ a Fraïssé class in language $\mathcal{L}$, and $\mathcal{K}^<$ the ordered expansion of $\mathcal{K}$. If $\mathcal{K}$ is an unrestricted Fraïssé class with free amalgamation, then $\mathcal{K}$ has SFAP and $\mathcal{K}^<$ has SDAP. If $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ for some set $\mathcal{F}$ of finite irreducible and 3-irreducible $\mathcal{L}$-structures, then $\mathcal{K}$ has SFAP and $\mathcal{K}^<$ has SDAP. All such classes have Fraïssé limits satisfying SDAP$^+$. Hence, the Fraïssé limits of all such classes admit big Ramsey structures, and their exact big Ramsey degrees have a simple characterization.*

We now discuss previous results recovered by Theorem 3.6, as well as their original proof methods.

In [30], Laflamme, Sauer, and Vuksanovic characterized the exact big Ramsey degrees of the Rado graph, generic directed graph, and generic tournament. More generally, they characterized exact big Ramsey degrees for the Fraïssé limit of any unrestricted Fraïssé class in a language with finitely many binary relations (which they called "universal structures" and which were called "simple structures" in [15]). Their characterization is exactly recovered in our Theorem 6.7. Their proof utilized Milliken's theorem for strong trees [35] and the method of envelopes, building on exact upper bound results for big Ramsey degrees of the Rado graph due to Sauer in [42]. The Fraïssé classes in their collection of "universal structures" with free amalgamation have SFAP, so Theorem 3.6 recovers their results for these classes. Further, all of their "universal structures" satisfy SDAP$^+$; the Main Theorem recovers their characterization of the exact big Ramsey degrees for these classes of structures.

For each $k \geq 3$, the Fraïssé class of finite k-uniform hypergraphs has SFAP, so Theorem 3.6 applies. Upper bounds on big Ramsey degrees for the generic 3-uniform hypergraph were proved by Balko, Chodounský, Hubička, Konečný, and Vena in [3]. Their methods are purely combinatorial, using the product Milliken theorem with the new development of auxiliary trees of matrices to keep track of the higher arity relations. They have independently developed this method to prove upper bounds for the generic k-uniform hypergraphs, for all $k \geq 3$, which is in preparation [4].

Lastly, Theorem 3.6 extends a result of El-Zahar and Sauer [17], in which they proved indivisibility for free amalgamation classes of k-uniform hypergraphs ($k \geq 3$) with forbidden 3-irreducible substructures. As these structures have only one isomorphism type of singleton substructure, their result says that for any $k \geq 3$ and any collection $\mathcal{F}$ of irreducible, 3-irreducible k-uniform hypergraphs, vertices in $\mathrm{Forb}(\mathcal{F})$ have big Ramsey degree one.

Theorem 3.6 recovers all the results highlighted in the previous three paragraphs and extends them to characterize the exact big Ramsey degrees and describe their big Ramsey structures.

We mention that for each $n \geq 2$, the Fraïssé class of finite n-partite graphs is easily seen to satisfy SFAP. John Howe proved in his PhD thesis [21] that the generic bipartite graph has finite big Ramsey degrees; his methods use an adjustment of Milliken's theorem. Finite big Ramsey degrees for n-partite graphs for all $n \geq 2$ follow from the more recent work of Zucker in [47]; his methods use a flexible version of coding trees and envelopes, but lower bounds are not attempted in that paper.

Next we consider disjoint amalgamation classes which are "$\mathbb{Q}$-like" in that their resemblance to linear orders makes them in some sense rigid enough to satisfy SDAP. Starting with the rationals as a linear order $(\mathbb{Q}, <)$, we shall show that the Fraïssé class of finite linear orders satisfies SDAP, and that $(\mathbb{Q}, <)$ satisfies SDAP$^+$. Further, the rational linear order with a vertex partition into finitely many dense pieces satisfies SDAP$^+$. We obtain a hierarchy of linear orders with nested convexly ordered equivalence relations that each satisfy SDAP$^+$. Lastly, we will show that the four non-trivial reducts of the rationals satisfy SDAP$^+$.

Given $n \geq 1$, let $\mathcal{LO}_n$ denote the Fraïssé class of finite structures with n-many independent linear orders. The language for $\mathcal{LO}_n$ is $\{<_i : i < n\}$, with each $<_i$ a binary relation symbol. In standard notation, $\mathcal{LO}$ denotes $\mathcal{LO}_1$.

Proposition 3.7. *The Fraïssé limit of $\mathcal{LO}$, namely the rational linear order, satisfies SDAP$^+$. For each $n \geq 2$, $\mathcal{LO}_n$ satisfies SDAP.*

Proof. Fixing $n \geq 1$, suppose $\mathbf{A}$ and $\mathbf{C}$ are in $\mathcal{LO}_n$ with $\mathbf{A}$ a substructure of $\mathbf{C}$ and $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$. Let $\mathbf{C}'$ be the extension of $\mathbf{C}$ by one vertex, a', satisfying the following: For each $i < n$, if $v <_i w$ in $\mathbf{C}$, then $v <_i a'$ and $a' <_i w$ are in $\mathbf{C}'$; otherwise, $w <_i a'$ and $a' <_i v$ are in $\mathbf{C}'$. Define $\mathbf{A}'$ to be the induced substructure $\mathbf{C}' \restriction (\mathrm{A} \cup \{a'\})$ of $\mathbf{C}'$.

Suppose that $\mathbf{B}$ is a finite linear order containing $\mathbf{A}'$ as a substructure, and let σ and τ be 1-types over $\mathbf{B}$ with the property that $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w/\mathbf{A}')$. Suppose that $\mathbf{D}$ is a one-vertex extension of $\mathbf{B}$ by the vertex v' so that $\mathrm{tp}(v'/\mathbf{B}) = \sigma$ holds. Now let $\mathbf{E}$ be an extension of $\mathbf{D}$ by one vertex w' satisfying $\mathrm{tp}(w'/\mathbf{B}) = \tau$. For each $i < n$, $v <_i w$ holds in $\mathbf{C}'$ if and only if $x <_i a'$ is in σ and $a' <_i x$ is in τ. (The opposite, $w <_i v$, holds in $\mathbf{C}'$ if and only if $a' <_i x$ is in σ and $x <_i a'$ is in τ.) It follows that $v' <_i w'$ holds in $\mathbf{E}$ if and only if $v <_i w$ holds in $\mathbf{C}$. Therefore, we automatically obtain $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$. Thus, SDAP holds.

Lemma 4.32 will show that $\mathcal{LO}$ satisfies the Diagonal Coding Tree Property, and the Extension Property will trivially hold. Hence, $\mathcal{LO}$ will satisfy SDAP$^+$. □

Next, we consider Fraïssé classes of structures with a linear order and a finite vertex partition of bounded size. Following the notation in [29], for each $n \geq 2$, let $\mathcal{P}_n$ denote the Fraïssé class with language $\{<, P_1, \dots, P_n\}$, where $<$ is a binary relation symbol and each P_i a unary relation symbol, such that in any structure in $\mathcal{P}_n$, $<$ is interpreted as a linear order and the interpretations of the P_i partition the vertices. The Fraïssé limit of $\mathcal{P}_n$, denoted by $\mathbb{Q}_n$, is the rational linear order with a partition of its underlying set into n definable pieces, each of which is dense in $\mathbb{Q}$.

Proposition 3.8. *For each $n \geq 1$, the Fraïssé limit $\mathbb{Q}_n$ of the Fraïssé class $\mathcal{P}_n$ satisfies SDAP$^+$.*

Proof. The proof is almost exactly the same as that for the rationals. Fixing $n \geq 1$, suppose $\mathbf{A}$ and $\mathbf{C}$ are in $\mathcal{P}_n$ with $\mathbf{A}$ a substructure of $\mathbf{C}$ and $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$. Let $\mathbf{C}'$ be the extension of $\mathbf{C}$ by one vertex, a', such that $v < w$ in $\mathbf{C}$ if and only if $v < a'$ and $a' < w$ in $\mathbf{C}'$; (otherwise, $w < v$ and $w < a'$ and $a' < v$ hold in $\mathbf{C}'$). Let $\mathbf{A}' = \mathbf{C}' \restriction (\mathrm{A} \cup \{a'\})$.

Given any $\mathbf{B}, \sigma, \tau, \mathbf{D}, v''$ as in (2) and (3) of Part (B) of Definition 2.10, any extension of $\mathbf{D}$ by one vertex w'' to a structure $\mathbf{E}$ with $\mathrm{tp}(w''/\mathbf{B}) = \tau$ automatically has $v'' < w''$ holding in $\mathbf{E}$ if and only if $v' < a' < w'$ holds in $\mathbf{A}'$. Since each P_i is a unary relation, $P_i(x)$ is in σ if and only if $P_i(v')$ holds. Thus, it follows that $P_i(v'')$ holds in $\mathbf{E}$ for that i such that $P_i(v)$ holds in $\mathbf{A}$. Likewise for w''. Therefore, $\mathbf{E} \restriction (\mathrm{A} \cup \{v'', w''\}) \cong \mathbf{C}$. Thus, SDAP holds.

The Extension Property trivially holds for $\mathbb{Q}_n$. SDAP$^+$ follows from the fact that the coding tree of 1-types for $\mathbb{Q}_n$ is a skew tree with splitting degree two, from which the construction of a tree satisfying SDAP$^+$ will easily follow (see Lemma 4.30). □

Next, we consider Fraïssé classes with a linear order and finitely many convexly ordered equivalence relations: An equivalence relation on a linearly ordered set is *convexly ordered* if each of its equivalence classes is an interval with respect to the linear order.

Given the language $\mathcal{L} = \{<, E\}$, where $<$ and E are binary relation symbols, let $\mathcal{COE}$ denote the Fraïssé class of *convexly ordered equivalence relations*, $\mathcal{L}$-structures in which $<$ is interpreted as a linear order and E as an equivalence relation that is convex with respect to that order. The Fraïssé limit of $\mathcal{COE}$, denoted by $\mathbb{Q}_{\mathbb{Q}}$, is the dense linear order without endpoints with an equivalence relation that has infinitely many equivalence classes, each an interval of order-type $\mathbb{Q}$, and with an induced order on the set of equivalence classes that is also of order-type $\mathbb{Q}$. One can think of $\mathbb{Q}_{\mathbb{Q}}$ as $\mathbb{Q}$ copies of $\mathbb{Q}$ with the lexicographic order. This structure was described by Kechris, Pestov, and Todorcevic in [25], where they proved that its automorphism groups is extremely amenable; from the main result of [25], it then follows that $\mathcal{COE}$ has the Ramsey property. This generated interest in the question of whether $\mathbb{Q}_{\mathbb{Q}}$ has finite big Ramsey degrees or big Ramsey structures.

Let $\mathcal{COE}_2$ denote the Fraïssé class in language $\{<, E_0, E_1\}$, where $<$, E_0 and E_1 are binary relation symbols, such that in any structure in $\mathcal{COE}_2$, $<$ is interpreted as a linear order, E_0 and E_1 as convexly ordered equivalence relations, and with the additional property that the interpretation of E_1 is a coarsening of that of E_0; that is, for any $\mathbf{A}$ in $\mathcal{COE}_2$, $a\, E_0^{\mathbf{A}}\, b$ implies $a\, E_1^{\mathbf{A}}\, b$. Then $\mathrm{Flim}(\mathcal{COE}_2)$ is $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$, that is $\mathbb{Q}$ copies of $\mathbb{Q}_{\mathbb{Q}}$; we shall denote this as $(\mathbb{Q}_{\mathbb{Q}})_2$. One can see that this recursive construction gives rise to a hierarchy of dense linear orders without endpoints with finitely many convexly ordered equivalence relations, where each successive equivalence relation coarsens the previous one. In general, let $\mathcal{COE}_n$ denote the Fraïssé class in the language $\{<, E_0, \dots, E_{n-1}\}$ where $<$ is interpreted as a linear order and each E_i ($i < n$) is interpreted as a convexly ordered equivalence relation, and such that for each $i < n-2$, the interpretation of E_{i+1} coarsens that of E_i. Let $(\mathbb{Q}_\mathbb{Q})_n$ denote the Fraïssé limit of $\mathcal{COE}_n$.

More generally, we may consider Fraïssé classes that are a blend of the $\mathcal{COE}_n$ and $\mathcal{P}_p$, having finitely many linear orders, finitely many convexly ordered equivalence relations, and a partition into finitely many pieces (each of which, in the Fraïssé limit, will be dense). Let $\mathcal{L}_{m,n,p}$ denote the language consisting of finitely many binary relation symbols, $<_0, \dots, <_{m-1}$, finitely many binary relation symbols $E_0, \dots, E_{n-1}$, and finitely many unary relation symbols $P_0, \dots, P_{p-1}$. A Fraïssé class $\mathcal{K}$ in language $\mathcal{L}_{m,n,p}$ is a member of $\mathcal{LOE}_{m,n,p}$ if each $<_i$, $i < m$, is interpreted as a linear order, each E_j, $j < n$, is interpreted as a convexly ordered equivalence relation with respect to exactly one of the linear orders $<_{i_j}$, for some $i_j < \ell$, and the interpretations of the P_k, $k < p$, induce a vertex partition into at most p pieces. Let $\mathcal{LOE}$ be the union over all triples (m, n, p) of $\mathcal{LOE}_{m,n,p}$. Let $\mathcal{COE}_{n,p}$ be the Fraïssé class in $\mathcal{LOE}_{1,n,p}$ for which the reduct to the language $\{<_0, E_0, \dots, E_{n-1}\}$ is a member of $\mathcal{COE}_n$.

Proposition 3.9. *Each Fraïssé class in $\mathcal{LOE}$ satisfies SDAP. Moreover, for any n, p, the Fraïssé limit of $\mathcal{COE}_{n,p}$ satisfies SDAP$^+$.*

Proof. Suppose $\mathbf{A}$ and $\mathbf{C}$ are in $\mathcal{K}$ with $\mathbf{A}$ a substructure of $\mathbf{C}$ and $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$. The unary relations are handled exactly as they were in Proposition 3.8, so we need to check that SDAP holds for the binary relations.

Let $\mathbf{C}'$ be an extension of $\mathbf{C}$ by vertices a'_k ($k < m+n$) satisfying the following: For each $i < m$, $v <_i w$ if and only if $v <_i a'_i$ and $a'_i <_i w$ in $\mathbf{C}'$. Given $j < n$, if $v \, E_j \, w$ holds in $\mathbf{C}$, then require that a'_{m+j} satisfies $v \, E_j \, a'_{m+j}$ and $w \, E_j \, a'_{m+j}$ in $\mathbf{C}'$. If $v \not\!E_j \, w$ holds in $\mathbf{C}$, then require that a'_{m+j} satisfies $v \, E_j \, a'_{m+j}$ and $w \not\!E_j \, a'_{m+j}$ in $\mathbf{C}'$. Let $\mathbf{A}' = \mathbf{C}' \restriction (\mathrm{A} \cup \{a'_k : k < m+n\})$.

Suppose that $\mathbf{B} \in \mathcal{K}$ contains $\mathbf{A}'$ as a substructure, and let σ and τ be consistent realizable 1-types over $\mathbf{B}$ with the property that $\sigma \restriction \mathbf{A}' = \mathrm{tp}(v/\mathbf{A}')$ and $\tau \restriction \mathbf{A}' = \mathrm{tp}(w/\mathbf{A}')$. Suppose that $\mathbf{D}$ is a one-vertex extension of $\mathbf{B}$ by the vertex v' satisfying $\mathrm{tp}(v'/\mathbf{B}) = \sigma$. Now let $\mathbf{E}$ be an extension of $\mathbf{D}$ by one vertex w' satisfying $\mathrm{tp}(w'/\mathbf{B}) = \tau$. The same argument as in the proof of Proposition 3.7 ensures that for each $i < m$, $v' <_i w'$ in $\mathbf{E}$ if and only if $v <_i w$ in $\mathbf{C}$.

Fix $j < n$. If $v \, E_j \, w$ in $\mathbf{C}$, then as $v \, E_j \, a'_{m+j}$ and $w \, E_j \, a'_{m+j}$ hold in $\mathbf{C}'$, the formula $x \, E_j \, a'_{m+j}$ is in both σ and τ. Since v' satisfies σ and w' satisfies τ, it follows that $v \, E_j \, w$ in $\mathbf{E}$. On the other hand, if $v \not\!E_j \, w$ holds in $\mathbf{C}$, then the formula $x \, E_j \, a'_{m+j}$ is in σ and $x \not\!E_j \, a'_{m+j}$ is in τ. Again, since v' satisfies σ and w' satisfies τ, it follows that $v \not\!E_j \, w$ in $\mathbf{E}$. Thus, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$. Hence SDAP holds.

Given any n, p, we will show that the Fraïssé limit of $\mathcal{COE}_{n,p}$ satisfies the Diagonal Coding Tree Property in Lemma 4.32 and that the Extension Property holds in Lemma 5.8, ensuring that SDAP$^+$ holds. □

This brings us to our second collection of big Ramsey structures.

Theorem 3.10. *The following Fraïssé structures satisfy SDAP$^+$. Hence they admit big Ramsey structures.*

(1) The rationals, $\mathbb{Q}$*.*
(2) $\mathbb{Q}_n$*, for each* $n \geq 1$*.*
(3) $\mathbb{Q}_{\mathbb{Q}}$*, and more generally,* $(\mathbb{Q}_{\mathbb{Q}})_n$ *for each* $n \geq 2$*.*
(4) The Fraïssé limit of any Fraïssé class in $\mathcal{COE}_{n,p}$*, for any* $n, p \geq 1$*.*

Proof. This follows from Propositions 3.7, 3.8 and 3.9, and the Main Theorem. □

We now discuss previous results which are recovered in Theorem 3.10, and results which are new.

Part (1) of Theorem 3.10 recovers the following previously known results: Upper bounds for finite big Ramsey degrees of the rationals were found by Laver [32] using Milliken's theorem. The big Ramsey degrees were characterized and computed by Devlin in [9]. Zucker interpreted Devlin's characterization into a big Ramsey structure, from which he then constructed the universal completion flow of the rationals in [46].

Exact big Ramsey degrees of the structures $\mathbb{Q}_n$ were characterized and calculated by Laflamme, Nguyen Van Thé, and Sauer in [29], using a colored level set version Milliken Theorem which they proved specifically for their application. The work in this paper using coding trees of 1-types provides a new way to view and recover their characterization of the big Ramsey degrees. From their work on $\mathbb{Q}_2$, Laflamme, Nguyen Van Thé, and Sauer further calculated the big Ramsey degrees of the circular directed graph $\mathbf{S}(2)$ in [29]. Exact Ramsey degrees of $\mathbf{S}(n)$ for all $n \geq 3$ were recently calculated by Barbosa in [5] using category theory methods. These structures $\mathbf{S}(n)$ have ages which do not satisfy SDAP.

Part (3) of Theorem 3.10 answers a question posed by Zucker during the open problem session at the 2018 BIRS Workshop on *Unifying Themes in Ramsey Theory*: He asked whether $\mathbb{Q}_{\mathbb{Q}}$ has finite big Ramsey degrees and whether it admits a big Ramsey structure. At that meeting, proofs that $\mathbb{Q}_{\mathbb{Q}}$ has finite big Ramsey degrees were found by Hubička using unary functions and strong trees, by Zucker using similar methods, and by Dobrinen using an approach that involved developing a topological Ramsey space with strong trees as bases, where each node in the given base is replaced with a strong tree. None of these proofs have been published, nor were those upper bounds shown to be exact. The result in this paper via SDAP$^+$ and coding trees of 1-types characterizes exact big Ramsey degrees and proves that $\mathbb{Q}_{\mathbb{Q}}$ admits a big Ramsey structure, and moreover, shows how it fits into a broader scheme of structures which have easily described big Ramsey degrees.

Part (4) of Theorem 3.10 in its full generality is new.

Lastly, we consider the reducts of the rational linear order. There are five such reducts up to interdefinability. These are the trivial countably infinite structure $\mathbb{Q}$ (with no relations), the rational linear order $(\mathbb{Q}, <)$ itself, and the Fraïssé structures that have one of the following relations, each definable from $(\mathbb{Q}, <)$: The ternary linear *betweenness* relation B, where

$$B(a,b,c) \iff (a < b < c) \vee (c < b < c); \tag{5}$$

the ternary *circular order* relation K, where

$$K(a,b,c) \iff (a < b < c) \vee (b < c < a) \vee (c < a < b); \tag{6}$$

and the quaternary *separation* relation S, where

$$\begin{aligned} S(a,b,c,d) \iff & (K(a,b,c) \wedge K(b,c,d) \wedge K(c,d,a)) \\ & \vee (K(d,c,b) \wedge K(c,b,a) \wedge K(b,a,d)). \end{aligned} \tag{7}$$

We shall refer to these as the "the five reducts of the rational linear order", and the latter four as the non-trivial ones.

Proposition 3.11. *The four non-trivial reducts of the rational linear order satisfy SDAP*$^+$.

Proof. Proposition 3.7 showed that the class of finite linear orders $\mathcal{LO}$, satisfies SDAP. A similar argument shows that the betweenness relation satisfies SDAP: Let $\mathcal{B}$ denote the age of the reduct of the rationals with the betweenness relation B. Suppose we are given $\mathbf{A}$ and $\mathbf{C}$ in $\mathcal{B}$ such that $\mathbf{A}$ is a substructure of $\mathbf{C}$ and $\mathrm{C} \setminus \mathrm{A} = \{v, w\}$. Extend $\mathbf{C}$, if necessary, to a structure $\mathbf{C}'$ so that for any $a \in \mathrm{A}$, the following hold:

(1) If $B(a, v, w)$ holds in $\mathbf{C}$, then there are some $a', b' \in \mathrm{C}'$ such that $B(a, a', b')$, $B(a, v, a')$, $B(v, a', b')$, and $B(a', b', w)$ hold in $\mathbf{C}'$.
(2) If $B(v, w, a)$ holds in $\mathbf{C}$, then there are some $a', b' \in \mathrm{C}'$ such that $B(a', b', a)$, $B(v, a', b')$, $B(a', b', w)$, and $B(a', b', w)$ hold in $\mathbf{C}'$.
(3) If $B(v, a, w)$ holds in $\mathbf{C}$, then there is some $a' \in \mathrm{C}'$ such that $B(v, a, a')$ and $B(a, a', w)$ hold in $\mathbf{C}'$.

Similarly, if the roles of v and w are reversed in the above three cases. Let $\mathbf{A}' = \mathbf{C}' \restriction (\mathrm{C}' \setminus \{v, w\})$.

Let $\mathbf{B}$ extend $\mathbf{A}'$, and let complete realizable 1-types σ, τ over $\mathbf{B}$ such that $\mathrm{tp}(v/\mathbf{A}') = \sigma \restriction \mathbf{A}'$ and $\mathrm{tp}(w/\mathbf{A}') = \tau \restriction \mathbf{A}'$ be given. Then given a one-vertex extension $\mathbf{D}$ of $\mathbf{B}$ by a vertex v' satisfying $\mathrm{tp}(v'/\mathbf{B}) = \sigma$, and a one-vertex extension $\mathbf{E}$ of $\mathbf{D}$ by a vertex w' satisfying $\mathrm{tp}(w'/\mathbf{B}) = \tau$ the following holds: If $B(a, v, w)$ holds in $\mathbf{C}$, then by (1), the formulas $B(a, x, a')$ and $B(x, a', b')$ are in σ, and the formula $B(a', b', x)$ is in τ. Therefore, $B(a, a', b')$, $B(a, v', a')$, $B(v', a', b')$ and $B(a', b', w')$ hold in $\mathbf{E}$. Thus, $B(a, v', w')$ must hold in $\mathbf{E}$; hence, $\mathbf{E} \restriction (\mathrm{A} \cup \{v', w'\}) \cong \mathbf{C}$. Hence, SDAP holds.

For the ternary circular order relation K, given $\mathbf{A}$ and $\mathbf{C}$, finding a structure $\mathbf{A}'$ such that the two given types are $\mathbf{C}$-interpolated in $\mathbf{A}'$ over $\mathbf{A}$ is even simpler: If $K(a, v, w)$ holds in $\mathbf{C}$ for some $a \in \mathrm{A}$, take $\mathbf{C}'$ with some vertex a' satisfying $K(a, v, a')$ and $K(a', w, a)$. If $K(v, a, w)$ holds for some $a \in \mathrm{A}$, then take $\mathbf{C}'$ with some vertex a' satisfying $K(a', v, a)$ and $K(a, w, a')$. Then letting $\mathbf{A}' = \mathbf{C}' \restriction (\mathrm{C}' \setminus \{v, w\})$, by a similar argument as above, the conclusion of SDAP holds.

Finally, we consider the ternary separation relation, S. Suppose that $S(a_0, a_1, v, w)$ holds in $\mathbf{C}$, for some $a_0, a_1 \in \mathrm{A}$. Then make sure that $\mathbf{C}'$ has some vertex a' so that $S(a_0, a_1, v, a')$ and $S(a', w, a_0, a_1)$ hold in $\mathbf{C}'$, and let $\mathbf{A}' = \mathbf{C}' \restriction (\mathrm{C}' \setminus \{v, w\})$. Then given any $\mathbf{B}, \sigma, \tau, \mathbf{D}$ as in the hypotheses of SDAP, with $\{v'\} = \mathrm{D} \setminus \mathrm{B}$, any 1-point extension $\mathbf{E}$ of $\mathbf{D}$ with $\mathrm{tp}(w'/\mathbf{B}) = \tau$ will satisfy $S(a_0, a_1, v', w')$.

That SDAP$^+$ holds will follow easily from the fact that the coding trees of 1-types look exactly like that of $\mathbb{Q}$ after the first two or three levels, that is, they are skew and binary splitting on all but finitely many levels. (See Lemma 4.30.) The Extension Property trivially holds, as it does for $\mathbb{Q}$. □

Big Ramsey degrees for the trivial countably infinite structure are all equal to one, by Ramsey's Theorem. Proposition 3.11, Ramsey's Theorem, and the Main Theorem thus yield the following.

Theorem 3.12. *Each of the five reducts of the rational linear order admits a big Ramsey structure.*

A recent result of Mašulović (Corollary 7.2 in [34]) uses category-theoretic methods to show that all five reducts of the rationals have finite big Ramsey degrees. Our approach provides a new way to view these reducts via coding trees of 1-types. See Figure 1 and surrounding discussion in Section 4 for further details.

We now turn to Fraïssé classes that do not fall within the purview of this paper. The sorts of Fraïssé classes which we know do not satisfy SDAP are those with some forbidden irreducible substructure which is not 3-irreducible. For instance, the ages of the k-clique-free Henson graphs, most metric spaces, and the generic partial order do not satisfy SDAP. We present two concrete examples of Fraïssé classes failing SDAP to give an idea of how failure arises from interplays of two vertices.

Example 3.13 (SFAP fails for triangle-free graphs)**.** Let $\mathcal{G}_3$ denote the Fraïssé class of finite triangle-free graphs. Let $\mathbf{A}$ be the graph with two vertices $\{a_0, a_1\}$ forming a non-edge, and let $\mathbf{C}$ be the graph with vertices $\{a_0, a_1, v, w\}$ with exactly one edge, $v\,E\,w$. Suppose $\mathbf{B}$ has vertices $\{a_0, a_1, b\}$, where $b \notin \{v, w\}$. Let $\sigma = \{\neg E(x, a_0) \wedge \neg E(x, a_1) \wedge E(x, b)\}$ and $\tau = \{\neg E(x, a) \wedge E(x, a_1) \wedge E(x, b)\}$. Then $\sigma \restriction \mathbf{A} = \mathrm{tp}(v/\mathbf{A})$, $\tau \restriction \mathbf{A} = \mathrm{tp}(w/\mathbf{A})$, and $\sigma \neq \tau$.

Suppose $\mathbf{E} \in \mathcal{G}_3$ is a graph satisfying the conclusion of Definition 2.8. To simplify notation, suppose that $\mathbf{E}$ has universe $\mathrm{E} = \{a_0, a_1, b, v, w\}$, with the obvious inclusion maps being the amalgamation maps. Then $\mathrm{tp}(v/\mathbf{B}) = \sigma$, $\mathrm{tp}(w/\mathbf{B}) = \tau$, and $\mathbf{E} \restriction \{a_0, a_1, v, w\} \cong \mathbf{C}$, so each pair in $\{b, v, w\}$ has an edge in $\mathbf{E}$. But this implies that $\mathbf{E}$ has a triangle, contradicting $\mathbf{E} \in \mathcal{G}_3$. Therefore, SFAP fails for $\mathcal{G}_3$.

The failure of SDAP for partial orders can be proved similarly, by taking $\mathbf{C}$ to have two vertices not in $\mathbf{A}$ which are unrelated to each other, and constructing $\mathbf{B}$, σ, τ so that any extension $\mathbf{E}$ satisfying σ and τ induces a relation between any v', w' satisfying σ, τ respectively, in such a way that transitivity forces there to be a relation between v' and w'.

We now give an example where SFAP fails in a structure with a relation of arity higher than two.

Example 3.14 (SFAP fails for 3-hypergraphs forbidding the irreducible 3-hypergraph on four vertices with three hyper-edges)**.** Suppose our language has one ternary relation symbol R. Let $\mathbf{I}$ denote a "pyramid", the structure on four vertices with exactly three hyper-edges; that is, say $\mathrm{I} = \{i, j, k, \ell\}$ and $\mathbf{I}$ consists of the relation $\{R^{\mathbf{I}}(i, j, k), R^{\mathbf{I}}(i, j, \ell), R^{\mathbf{I}}(i, k, \ell)\}$. Then every two vertices in I are in some relation in $\mathbf{I}$, so $\mathbf{I}$ is irreducible. However, the triple $\{j, k, \ell\}$ is not contained in any relation in $\mathbf{I}$.

The free amalgamation class $\mathrm{Forb}(\{\mathbf{I}\})$ does not satisfy SFAP: Let $\mathbf{A}$ be the singleton $\{a\}$, with $R^{\mathbf{A}} = \emptyset$, and let $\mathbf{C}$ have universe $\{a, c_0, c_1\}$ with $R^{\mathbf{C}} = \{(a, c_0, c_1)\}$. Let $\mathbf{B}$ have universe $\{a, b\}$, and let σ and τ both be the 1-types $\{R(x, a, b)\}$ over $\mathbf{B}$. Suppose that $\mathbf{E} \in \mathrm{Forb}(\{\mathbf{I}\})$ satisfies the conclusion of Definition 2.10. Then $\mathbf{E}$ has universe $\{a, b, c_0, c_1\}$ and $R^{\mathbf{E}} = \{(a, c_0, c_1), (c_0, a, b), (c_1, a, b)\}$. Hence $\mathbf{E}$ contains a copy of $\mathbf{I}$, contradicting that $\mathbf{E} \in \mathrm{Forb}(\{\mathbf{I}\})$.

Remark 3.15. The same argument shows that SFAP fails for any free amalgamation class $\mathrm{Forb}(\mathcal{F})$ where some $\mathbf{F} \in \mathcal{F}$ is not 3-irreducible. This does not imply that Fraïssé limits of such classes fail to have finite big Ramsey degrees; only that proving existence of such degrees will require methods more closely related to those in [13], [11], and [47] which can handle binary structures with forbidden substructures that are irreducible but not 3-irreducible, combined with methods in this paper or in [3] which handle higher arity relations. It surely must be the case that the generic 3-hypergraph omitting the "pyramid", the structure on four vertices with three 3-hyperedges, has finite big Ramsey degrees and admits a big Ramsey structure. Indeed, it seems likely that all free amalgamation classes with finitely many finite forbidden substructures should have finite big Ramsey degrees. It is simply that proving this will involve a hybrid of methods in this paper and those developed for Fraïssé classes in binary relational languages with forbidden irreducible substructures.

We now present a catalogue of many (though not all) of the known results regarding finite big Ramsey degrees (upper bounds) and characterizations of exact big Ramsey degrees (canonical partitions). A blank box means the property has not yet been proved or disproved. All previously known results for Fraïssé classes with Fraïssé limits satisfying SDAP$^+$ are recovered by our Theorem 6.7. New results in this paper are indicated by the number of the theorem from which they follow.

In all cases where exact big Ramsey degrees have been characterized, this has been achieved via finding canonical partitions. Moreover, these canonical partitions have been found in terms of similarity types of antichains in trees of 1-types, either explicitly or implicitly. Once one has such canonical partitions, the existence of a big Ramsey structure follows from Theorem 6.9 in conjunction with Zucker's Theorem 7.1 in [46]. Thus, we do not include a column for existence of big Ramsey structures.

Key

- DA: Disjoint Amalgamation
- FA: Free Amalgamation
- SDAP: strongest of SFAP, SDAP$^+$, or SDAP known to hold
- FBRD: Finite big Ramsey degree
- CP: Exact big Ramsey degrees characterized via Canonical Partitions

(1) $\mathbb{Q}$-like structures

Fraïssé limit	**DA**	**FA**	**SDAP**	**FBRD**	**CP**
$\mathbb{Q}$ with no relations	✓	✓	SDAP	[40]	[40]
$(\mathbb{Q}, <)$	✓	✗	SDAP$^+$	[32]	[9]
$\mathbb{Q}_n$	✓	✗	SDAP$^+$	[29]	[29]
$\mathbf{S}(2)$	✓	✗	✗	[29]	[29]
$\mathbf{S}(3)$,$\mathbf{S}(4)$, $\cdots$	✓	✗	✗	[5]	[5]
$\mathbb{Q}_{\mathbb{Q}}$, $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$, $\cdots$	✓	✗	SDAP$^+$	[Thm 5.22]	[Thm 6.7]
$(\mathbb{Q}, B)$,$(\mathbb{Q}, K)$,$(\mathbb{Q}, S)$	✓	✗	SDAP$^+$	[33]	[Thm 6.7]
Generic structures with two or more independent linear relations	✓	✗	SDAP	[22]	

(2) Unconstrained relational structures

Fraïssé limit	**DA**	**FA**	**SDAP**	**FBRD**	**CP**
Rado graph	✓	✓	SFAP	[42]	[42]
Generic two-graph	✓	✓	✗	[33]	
Generic bipartite	✓	✓	SFAP	[21]	[Thm 6.7]
Generic n-partite for $n \geq 3$	✓	✓	SFAP	[47]	[Thm 6.7]
Generic directed graph	✓	✓	SFAP	[30]	[30]
Generic tournament	✓	✗	SDAP$^+$	[30]	[30]
Generic unrestricted structures in a finite binary relational language	✓	✗	SDAP$^+$	[30]	[30]
Generic 3-uniform hypergraph	✓	✓	SFAP	[3]	[Thm 6.7]
Generic k-uniform hypergraph for $k > 3$	✓	✓	SFAP	[Thm 5.22] & [4]	[Thm 6.7]

(3) Constrained binary relational structures

Fraïssé limit	**DA**	**FA**	**SDAP**	**FBRD**	**CP**
Generic K_3-free graphs	✓	✓	✗	[13]	[1]
Generic K_n-free graphs for finite $n > 3$	✓	✓	✗	[11]	[1]
Generic structures in a finite binary relational language with finitely many finite irreducible forbidden substructures	✓	✓	✗	[47]	[1]
Generic poset	✓	✗	✗	[22]	

(4) Constrained higher arity relational structures

Fraïssé limit	**DA**	**FA**	**SDAP**	**FBRD**	**CP**
Generic pyramid-free 3-hypergraph	✓	✓	✗		
Fraïssé limit of Forb($\mathcal{F}$) where all $F \in \mathcal{F}$ are irreducible and 3-irreducible	✓	✓	SFAP	[Thm 5.22]	[Thm 6.7]
Fraïssé limit of Forb$(\mathcal{F})^<$, where all $F \in \mathcal{F}$ are irreducible and 3-irreducible	✓	✗	SDAP$^+$	[Thm 5.22]	[Thm 6.7]

Remark 3.16. Results on big Ramsey degrees of metric spaces appear in [22], [33], [34], and [38]: Nguyen Van Thé showed finite big Ramsey degrees for finite S-submetric spaces of ultrametric S-spaces [38], with S finite and nonnegative. Sauer established indivisibility (big Ramsey degree one for vertex colorings) of Urysohn S-metric spaces with S finite [43]. Mašulović proved finite big Ramsey degrees for Urysohn S-metric spaces, where S is a finite distance set with no internal jumps and a property called "compactness" in that paper, meaning that the distances are not too far apart. Recently, Hubička extended this to all Urysohn S-metric spaces where S is tight in addition to finite and nonnegative [22]. As SDAP fails for non-trivial metric spaces (for the same reason it fails for the triangle-free graphs and partial orders), we mention no details here.

4. Coding trees of 1-types for Fraïssé structures

Fix throughout a Fraïssé class $\mathcal{K}$ in a finite relational language $\mathcal{L}$. Recall that $\mathbf{K}$ denotes an *enumerated Fraïssé limit* for $\mathcal{K}$, meaning that $\mathbf{K}$ has universe ω. In order to avoid confusion, we shall usually use v_n instead of just n to denote the n-th member of the universe of $\mathbf{K}$, and we shall call this the *n-th vertex* of $\mathbf{K}$. For $n < \omega$, we write $\mathbf{K}_n$, and sometimes $\mathbf{K}\restriction n$, to denote the substructure of $\mathbf{K}$ on the set of vertices $\{v_i : i < n\}$. We call $\mathbf{K}_n$ an *initial segment* of $\mathbf{K}$. Note that $\mathbf{K}_0$ is the empty structure.

In Subsection 4.1, we present a general construction of trees of complete 1-types over initial segments of $\mathbf{K}$, which we call *coding trees*. Graphics of coding trees are then presented for various prototypical Fraïssé classes satisfying SDAP$^+$. In Subsection 4.2, we define *passing types*, extending the notion of *passing number* due to Laflamme, Sauer, and Vuksanovic in [30], which has been central to all prior results on big Ramsey degrees for binary relational structures. Then we extend the notion from [30] of *similarity type* for binary relational structures to structures with relations of any arity. In Subsection 4.3, we introduce *diagonal* coding trees. These will be key to obtaining precise big Ramsey degree results without appeal to the method of envelopes. We define the *Diagonal Coding Tree Property*, one of the conditions for SDAP$^+$ to hold, and then present the coding tree version of SDAP$^+$ (Definition 4.26), from which the formulation of SDAP in Definition 2.10 was extracted. We show how to construct diagonal subtrees representing copies of various Fraïssé structures, completing the proofs of the Diagonal Coding Tree Property in Theorem 3.6 and Propositions 3.7, 3.8, 3.9, and 3.11.

4.1. Coding trees of 1-types.

All types will be quantifier-free 1-types, with variable x, over some finite initial segment of $\mathbf{K}$. For $n \geq 1$, a type over $\mathbf{K}_n$ must contain the formula $\neg(x = v_i)$ for each $i < n$. Given a type s over $\mathbf{K}_n$, for any $i < n$, $s\restriction\mathbf{K}_i$ denotes the restriction of s to parameters from $\mathbf{K}_i$. Recall that the notation "tp" denotes a complete quantifer-free 1-type.

Definition 4.1 (The Coding Tree of 1-Types, $\mathbb{S}(\mathbf{K})$)**.** The *coding tree of 1-types* $\mathbb{S}(\mathbf{K})$ for an enumerated Fraïssé structure $\mathbf{K}$ is the set of all complete 1-types over initial segments of $\mathbf{K}$ along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ such that $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is simply inclusion.

We shall usually simply write $\mathbb{S}$, rather than $\mathbb{S}(\mathbf{K})$. Note that we make no requirement at this point on $\mathbf{K}$; an enumerated Fraïssé limit of any Fraïssé class (with no reference to its amalgamation or Ramsey properties) naturally induces a coding tree of 1-types as above. We say that $c(n)$ *represents* or *codes* the vertex v_n. Instead of writing $c(n)$, we shall usually write c_n for the n-th coding node in $\mathbb{S}$.

We let $\mathbb{S}(n)$ denote the collection of all 1-types $\mathrm{tp}(v_i/\mathbf{K}_n)$, where $i \geq n$. Note that each $c(n)$ is a node in $\mathbb{S}(n)$. The set $\mathbb{S}(0)$ consists of the 1-types over the empty structure $\mathbf{K}_0$. For $s \in \mathbb{S}(n)$, the immediate successors of s are exactly those $t \in \mathbb{S}(n+1)$ such that $s \subseteq t$. For each $n < \omega$, the set $\mathbb{S}(n)$ is finite, since the language $\mathcal{L}$ consists of finitely many finitary relation symbols.

We say that each node $s \in \mathbb{S}(n)$ has *length* $n+1$, and denote the length of s by $|s|$. Thus, all nodes in $\mathbb{S}$ have length at least one. While it is slightly unconventional to consider the roots of $\mathbb{S}$ as having length one, this approach lines up with the natural correspondence between nodes in $\mathbb{S}$ and certain sequences of partial 1-types that we define in the next paragraph. The reader wishing for a tree starting with a node of length zero may consider adding the empty set to $\mathbb{S}$, as this will have no effect on the results in this paper. A *level set* is a subset $X \subseteq \mathbb{S}$ such that all nodes in X have the same length.

Let $n < \omega$ and $s \in \mathbb{S}(n)$ be given. We let $s(0)$ denote the set of formulas in s involving no parameters; $s(0)$ is the unique member of $\mathbb{S}(0)$ such that $s(0) \subseteq s$. For $1 \leq i \leq n$, we let $s(i)$ denote the set of those formulas in $s \restriction \mathbf{K}_i$ in which v_{i-1} appears; in other words, the formulas in $s \restriction \mathbf{K}_i$ that are not in $s \restriction \mathbf{K}_{i-1}$. In this manner, each $s \in \mathbb{S}$ determines a unique sequence $\langle s(i) : i < |s|\rangle$, where $\{s(i) : i < |s|\}$ forms a partition of s. For $j < |s|$, $\bigcup_{i \leq j} s(i)$ is the node in $\mathbb{S}(j)$ such that $\bigcup_{i \leq j} s(i) \subseteq s$. For $\ell \leq |s|$, we shall usually write $s \restriction \ell$ to denote $\bigcup_{i<\ell} s(i)$.

Given $s, t \in \mathbb{S}$, we define the *meet* of s and t, denoted $s \wedge t$, to be $s \restriction \mathbf{K}_m$ for the maximum $m \leq \min(|s|, |t|)$ such that $s \restriction \mathbf{K}_m = t \restriction \mathbf{K}_m$. It can be useful to think of $s \in \mathbb{S}$ as the sequence $\langle s(0), \dots, s(|s|-1)\rangle$; then $s \wedge t$ can be interpreted in the usual way for trees of sequences.

It will be useful later to have specific notation for unary relations. We will let Γ denote $\mathbb{S}(0)$, the set of complete 1-types over the empty set that are realized in $\mathbf{K}$. If $\mathcal{L}$ has no unary relation symbols, then Γ will consist exactly of the "trivial" 1-type which is satisfied by every element of $\mathbf{K}$. For $\gamma \in \Gamma$, we write "$\gamma(v_n)$ holds in $\mathbf{K}$" when γ is the 1-type of v_n over the empty set; in practice, it will be the unary relation symbols in γ (if there are any) that will be of interest to us.

Remark 4.2. Our definition of $s(i)$ sets up for the definition of passing type in Subsection 4.2, which directly abstracts the notion of passing number used in [42] and [30], and in subsequent papers building on their ideas.

Remark 4.3. In the case where all relation symbols in the language $\mathcal{L}$ have arity at most two, the coding tree of 1-types $\mathbb{S}$ has bounded branching. If $\mathcal{L}$ has any relation symbol of arity three or greater, then $\mathbb{S}$ may have branching which increases as the levels increase. If such a Fraïssé class satisfies SDAP, sometimes more work still must be done in order to guarantee that its Fraïssé limit has SDAP$^+$.

We now provide graphics for coding trees of 1-types which are prototypical for the Fraïssé classes which we proved in Section 3 to have SDAP. We start with the rational linear order, since its coding tree of 1-types is the simplest, and also because the rationals were the first Fraïssé structure for which big Ramsey degrees were characterized (Devlin, [9]).

Example 4.4 (The coding tree of 1-types $\mathbb{S}(\mathbb{Q})$)**.** Figure 1 shows the coding tree of 1-types for $(\mathbb{Q}, <)$, the rationals as a linear order. This is the Fraïssé limit of $\mathcal{LO}$, the class of finite linear orders. We assume that the universe of $\mathbb{Q}$ is linearly ordered in order-type ω as $\langle v_n : n < \omega\rangle$. For each n, the coding node c_n is the 1-type of vertex v_n over the initial segment $\{v_i : i < n\}$ of $\mathbb{Q}$. (Recall that x is the variable in all of our 1-types.) Thus, the coding node c_0 is the empty 1-type, and c_1 is the 1-type $\{v_0 < x\}$. Thus, the coding nodes $\{c_0, c_1\}$ represent the linear order $v_0 < v_1$. Likewise, the coding node c_2 is the 1-type $\{x < v_0, x < v_1\}$ over the linear order $v_0 < v_1$. Hence, c_2 represents the vertex v_2 satisfying $v_2 < v_0 < v_1$. The

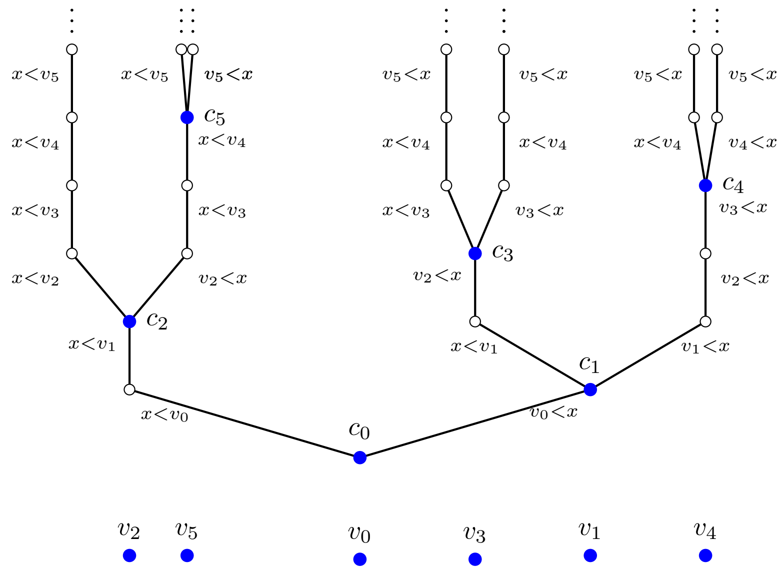

FIGURE 1. Coding tree of 1-types for $(\mathbb{Q}, <)$ and the linear order represented by its coding nodes.

coding node c_3 is the 1-type $\{v_0 < x, x < v_1, v_2 < x\}$, so c_3 represents the vertex v_3 satisfying $v_2 < v_0 < v_3 < v_1$. Below the tree, we picture the linear order on the vertices $v_0, \dots, v_5$ induced by the coding nodes. As the tree grows in height, the linear order represented by the coding nodes grows into the countable dense linear order with no endpoints.

Notice that only the coding nodes branch. This is because of the rigidity of the rationals: Given a non-coding node s on the same level as a coding node c_n (say $n \geq 1$), s is a 1-type which is satisfied by any vertex which lies in some interval determined by the vertices $\{v_i : i < n\}$, and v_n is not in that interval. Thus, the order between v_n and any vertex satisfying s is predetermined, so s does not split. Said another way, letting m denote the length of the meet of c_n and s, c_n and s must disagree on the formula $x < v_m$; hence, $x < v_m$ is in c_n if and only if $v_m < x$ is in s. In the case that the formula $x < v_m$ is in c_n, then it follows that $v_n < v_m$. On the other hand, any realization v_i of the 1-type s must satisfy $v_m < v_i$. Hence every realization of s by some vertex v_i must satisfy $v_n < v_i$. Thus, there is only one immediate successor of s in the tree of 1-types. The tree of 1-types for $\mathbb{Q}$ eradicates the extraneous structure which appears in the more traditional approach of using the full binary branching tree and Milliken's Theorem to approach big Ramsey degrees of the rationals.

Interestingly, the reducts of the rationals, $(\mathbb{Q}, B)$, $(\mathbb{Q}, K)$, and $(\mathbb{Q}, S)$ have coding trees of 1-types that look exactly like Figure 1 except for their first few levels, since the relations B and K are ternary and S is quaternary. After two or three levels, each coding node branches into exactly two immediate successors while the other nodes at that level do not branch. This has to do with the rigidity of these structures inherited as reducts of $(\mathbb{Q}, <)$.

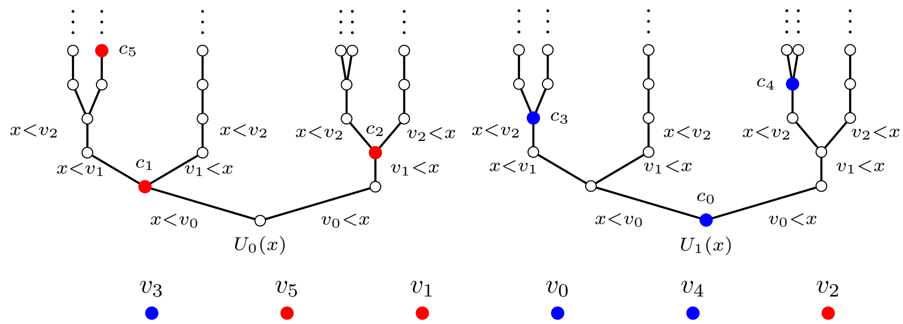

FIGURE 2. Coding tree of 1-types for $(\mathbb{Q}_2, <)$ and the linear order represented by its coding nodes.

Example 4.5 (The coding tree of 1-types $\mathbb{S}(\mathbb{Q}_2)$)**.** Next, we consider coding trees of 1-types for linear orders with equivalence relations with finitely many equivalence classes, each of which is dense in the linear order. Figure 2 provides a graphic for the coding tree of 1-types for the structure $\mathbb{Q}_2$, the rationals with an equivalence relation with two equivalence classes which are each dense in the linear order. This is the Fraïssé limit of $\mathcal{P}_2$ discussed just before Lemma 3.8. We point out that c_0 is the 1-type $\{U_1(x)\}$, c_1 is the 1-type $\{U_0(x), x < v_0\}$, c_2 is the 1-type $\{U_0(x), v_0 < x, v_1 < x\}$, etc.

Note that $\mathbb{S}(\mathbb{Q}_2)$ looks like two identical disjoint copies of a coding tree for $\mathbb{Q}$. This is because each of the two unary relations, representing the two equivalence classes, appears densely in the linear order. The ordered structure $\mathbb{Q}_2$ appears below the two trees as the vertices $v_0, v_1, \dots$. Unlike Figure 1 for $\mathbb{Q}$, the vertices in $\mathbb{Q}_2$ do not line up below the coding nodes in the trees representing them, since $\mathbb{S}(\mathbb{Q}_2)$ has two roots. However, if we modify our definition of coding tree of 1-types to have individual coding nodes c_n represent the unary relations satisfied by v_n (rather than $\mathbb{S}$ having $|\Gamma|$ many roots), this has the effect of producing a one-rooted tree with "γ-colored" coding nodes appearing cofinally in the tree, for each $\gamma \in \Gamma$. This approach then shows the linear order $\mathbb{Q}_2$ lining up below the coding nodes, recovers the characterization of the big Ramsey degrees in [29], and will aid us in proving SDAP$^+$ for $\mathcal{P}_2$. (See Definition 4.17 for this variation of tree of 1-types, which reproduces the approach in [29].) The formulation given in Definition 4.1 makes clear the distinction between exact big Ramsey degrees for Fraïssé classes with SFAP and those with SFAP plus a linear order, and sets up correctly for proving lower bounds in general.

Similarly, for any $n \geq 2$, $\mathbb{S}(\mathbb{Q}_n)$ will have n roots, and above each root, the n trees will be copies of each other.

Example 4.6 (The coding tree of 1-types $\mathbb{S}(\mathbb{Q}_{\mathbb{Q}})$)**.** Next, we present the tree of 1-types for the Fraïssé structure $\mathbb{Q}_{\mathbb{Q}}$. Recall that this is the Fraïssé limit of the class $\mathcal{CO}$ in the language $\mathcal{L} = \{<, E\}$, where $<$ is a linear order and E is a convexly ordered equivalence relation, meaning that all equivalence classes are intervals.

Figure 3 shows the first six levels of a coding tree of 1-types, $\mathbb{S}(\mathbb{Q}_{\mathbb{Q}})$. The formulas which are in the 1-types can be read from the graphic. For instance, c_0 is the empty type. c_1 is the 1-type $\{v_0 < x, E(x, v_0)\}$, so since the vertex v_1 satisfies this

1-type, we have $v_0 < v_1$ and $E(v_0, v_1)$ holding. Similarly, c_2 is the 1-type $\{x < v_0, \neg E(x, v_0), x < v_1, \neg E(x, v_1)\}$, so v_2 satisfies $v_2 < v_0$, and v_2 is not equivalent to either of v_0 or v_1. c_3 is the 1-type $\{v_0 < x, E(x, v_0), v_1 < x, E(x, v_1), v_2 < x, \neg E(x, v_2)\}$, and hence, we see that $v_1 < v_3$ and v_3 is equivalent to v_1 and hence also to v_0. Note that only coding nodes branch. Moreover, c_n has splitting degree two if c_n represents a vertex v_n which is equivalent to v_i for some $i < n$; otherwise c_n has splitting degree four. For each non-coding node s on the level of a coding node c_n, there is only one possible 1-type extending s over the initial structure on the first $n + 1$ vertices of $\mathbb{Q}_{\mathbb{Q}}$. We will show later in Lemma 4.32 that there is a coding subtree representing $\mathbb{Q}_{\mathbb{Q}}$ ensuring that SDAP$^+$ holds. The structures $\mathbb{Q}_{\mathbb{Q}_{\mathbb{Q}}}$, etc. have coding trees of 1-types which behave similarly.

In Figure 3, below the tree $\mathbb{S}(\mathbb{Q}_{\mathbb{Q}})$ is the linear order on the vertices $v_0, \dots, v_5$ represented by the coding nodes $c_0, \dots, c_5$; the lines between the vertices represent that they are in the same equivalence class. Thus, v_0, v_1, v_3 are all in the same equivalence class, v_2, v_4 are in a different equivalence class, and v_5 is in yet another equivalence class.

Next, we present graphics for coding trees of 1-types for some free amalgamation classes. The tree of 1-types for the Rado graph is simply a binary tree in which the coding nodes are dense and every node s at the level of the n-th coding node splits into two immediate successors, representing the two possible extensions of s to the 1-types $s \cup \{E(x, v_n)\}$ and $s \cup \{\neg E(x, v_n)\}$. This follows immediately from the Extension Property for the Rado graph. As this is simple to visualize, and as a graphic has already appeared in [14], we move on to bipartite graphs.

Example 4.7 (The coding tree of 1-types for the generic bipartite graph)**.** Figure 4 presents a coding tree of 1-types for the generic bipartite graph. The unary relations U_0 and U_1, which keep track of which partition each vertex is in, are represented by "red" and "blue", respectively. We have chosen to enumerate this structure so that odd indexed vertices are in one of the partitions, and even indexed vertices are in the other, for purely aesthetic reasons. The edge relation is represented as extension to the right, and non-edge is represented by extending left. On the left is the bipartite graph being represented by the coding nodes in the two-rooted tree of 1-types. For instance, c_0 is the 1-type $\{U_0(x)\}$, so v_0 is a vertex in the collection of "red" vertices. For another example, c_3 is the 1-type $\{U_1(x), E(x, v_0), \neg E(x, v_1), E(x, v_2)\}$. Thus, v_3 is in the collection of "blue" vertices and has edges exactly with v_0 and v_2. By Proposition 3.2, for each $n \geq 2$, the Fraïssé class of finite n-partite graphs satisfies SFAP.

Lastly, we consider free amalgamation classes with relations of higher arity. The prototypical example of this is the generic 3-uniform hypergraph, and discussing it should provide the reader with reasonable intuition about coding trees for higher arities.

Example 4.8 (The coding tree of 1-types for the generic 3-uniform hypergraph)**.** Figure 5 presents the coding tree of 1-types for the generic 3-uniform hypergraph. This tree has the property that every node at the same level branches into the same number of immediate successors, as there are no forbidden substructures. On the left of Figure 5 is a picture of the hypergraph being built, where v_n is the vertex satisfying the 1-type of the coding node c_n over the initial segment of the structure restricted to $\{v_i : i < n\}$.

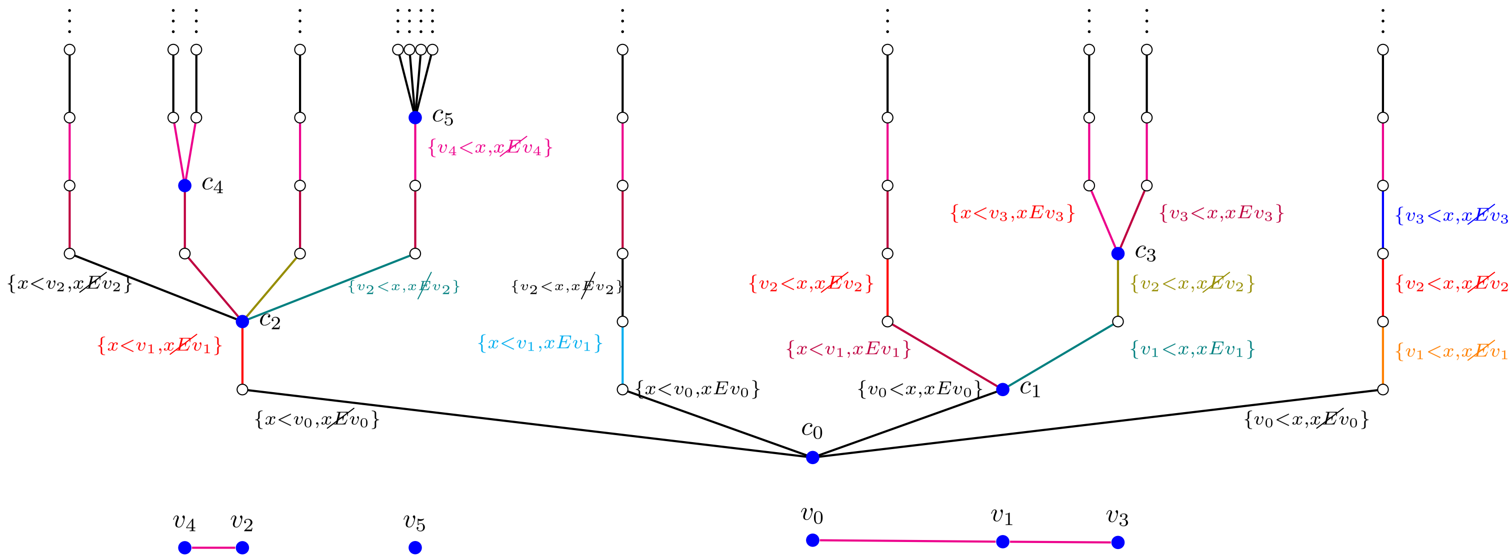

Figure 3. Coding tree of 1-types for $\mathbb{Q}_{\mathbb{Q}}$ and linear order with convex equivalence relations represented by its coding nodes.

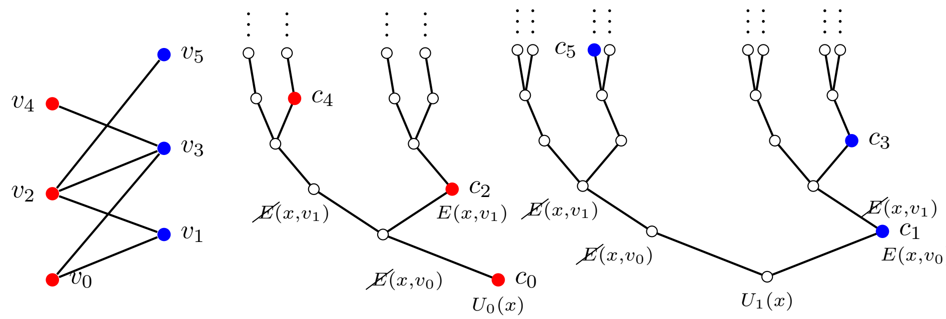

FIGURE 4. Coding tree of 1-types for the generic bipartite graph

Since hyperedges involve three vertices, c_0 and c_1 are both the empty 1-types. Letting R denote the 3-hyperedge relation, c_1 branches into two 1-types over $\{v_0, v_1\}$: $\{\neg R(x, v_0, v_1)\}$ and $\{R(x, v_0, v_1)\}$. Since $c_2 = \{R(x, v_0, v_1)\}$, it follows that $R(v_0, v_1, v_2)$ holds in the hypergraph represented on the left of the tree; this hyperedge is represented by the oval containing these three vertices.

Both nodes on the level of c_2 branch into four immediate successors. This is because for each node s at the level of c_2, the immediate successors of s range over the possibilities of adding a new formula $R(x, \cdot, \cdot)$ or $\neg R(x, \cdot, \cdot)$ containing the parameter v_2 and a choice of either v_0 or v_1 as the second parameter. In particular, the immediate successors of c_2 are the 1-types consisting of $\{R(x, v_0, v_1)\}$ unioned with one of the following:

(1) $\{\neg R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$;
(2) $\{\neg R(x, v_0, v_2), R(x, v_1, v_2)\}$;
(3) $\{R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$;
(4) $\{R(x, v_0, v_2), R(x, v_1, v_2)\}$.

Likewise, the immediate successors of the other node $s = \{\neg R(x, v_0, v_1)\}$ in level two of the tree consists of the extensions of s by one of the four above cases. In general, each node on the level of c_n branches into 2^n many immediate successors. This is because the new formulas in any immediate successor have the choice of $R(x, p, v_n)$ or its negation, where $p \in \{v_i : i < n\}$. However, the Fraïssé class of finite 3-uniform hypergraphs satisfies SFAP (by Proposition 3.2), and Theorem 4.28 will provide a skew subtree coding the generic 3-hypergraph in which the branching degree is two (that is, a diagonal subtree).

The coding node c_3 is the 1-type $\{\neg R(x, v_0, v_1), R(x, v_0, v_2), \neg R(x, v_1, v_2)\}$. Thus, the hypergraph being built on the left has the hyperedge $R(v_0, v_2, v_3)$. The coding node c_4 is the 1-type consisting of $R(x, v_0, v_1), R(x, v_1, v_2), R(x, v_2, v_3)$ along with $\neg R(x, p_0, p_1)$ where p_0, p_1 are parameters in $\{v_0, \dots, v_3\}$. This codes the new hyperedges $R(v_0, v_1, v_4)$, $R(v_1, v_2, v_4)$ and $R(v_2, v_3, v_4)$.

4.2. **Passing types and similarity.** As before, let $\mathbf{K}$ be an enumerated Fraïssé structure and $\mathbb{S} := \mathbb{S}(\mathbf{K})$ be the corresponding coding tree of 1-types. We begin by defining the notion of a subtree of $\mathbb{S}$. As is standard in Ramsey theory on infinite trees (see Chapter 6 of [45]), a subtree is not necessarily closed under initial

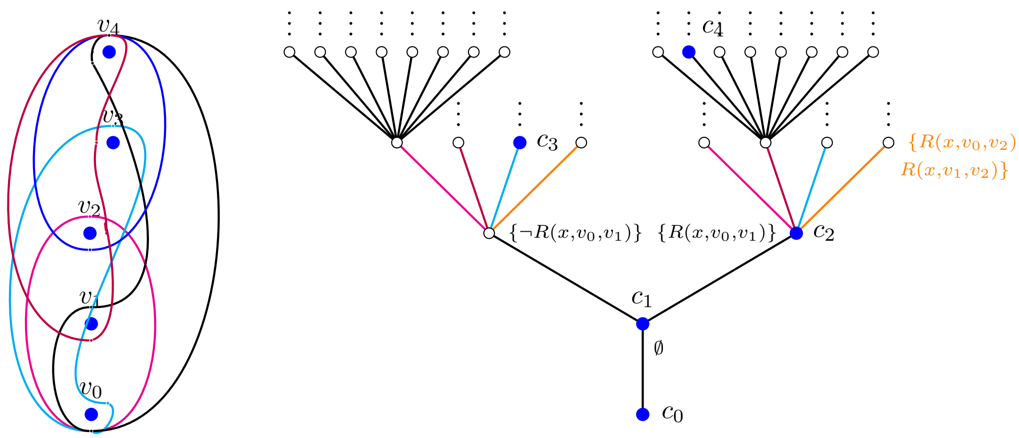

FIGURE 5. Coding tree of 1-types for the generic 3-uniform hypergraph.

segments, but rather it is closed under those portions of initial segments that have certain prescribed lengths.

Definition 4.9 (Subtree)**.** Let T be a subset of $\mathbb{S}$, and let L be the set of lengths of coding nodes in T and lengths of meets of two incomparable nodes (not necessarily coding nodes) in T. Then T is a *subtree* of $\mathbb{S}$ if T is closed under meets and closed under initial segments with lengths in L, by which we mean that whenever $\ell \in L$ and $t \in T$ with $\ell \leq |t|$, then $t \restriction \ell$ is also a member of T.

We now describe the natural correspondence from subtrees of $\mathbb{S}$ to substructures of $\mathbf{K}$. The following notation will aid in the translation.

Notation 4.10. Given a subtree $A \subseteq \mathbb{S}$, let $\langle c_n^A : n < N \rangle$ denote the enumeration of the coding nodes of A in order of increasing length, where $N \leq \omega$ is the number of coding nodes in A. Let

$$\mathrm{N}^A := \{i \in \omega : \exists m \, (c_i = c_m^A)\}, \tag{8}$$

the set of indices i such that c_i is a coding node in A. For $n < N$, let

$$\mathrm{N}_n^A := \{i \in \mathrm{N}^A : \exists m < n \, (c_i = c_m^A)\}, \tag{9}$$

the set of indices of the first n coding nodes in A. Recall that ω is the set of vertices for $\mathbf{K}$, and that we often use v_i to denote i, the i-th vertex of $\mathbf{K}$. Thus, N^A is precisely the set of vertices of $\mathbf{K}$ represented by the coding nodes in A. Let $\mathbf{K} \restriction A$ denote the substructure of $\mathbf{K}$ on universe N^A. We call this the *substructure of* $\mathbf{K}$ *represented by the coding nodes in* A, or simply *the substructure represented by* A.

The next definition extends the notion of *passing number* developed in [42] and [30] to code binary relations using pairs of nodes in regular splitting trees. Here, we extend this notion to relations of any arity.

Recall from the discussion after Definition 4.1 that for $s \in \mathbb{S}$, $s(0)$ denotes the set of formulas in s without parameters; and for for $1 \leq i < |s|$, $s(i)$ denotes the set of those formulas in $s \restriction \mathbf{K}_i$ in which v_{i-1} appears.

Definition 4.11 (Passing Type)**.** Given $s, t \in \mathbb{S}$ with $|s| < |t|$, we call $t(|s|)$ the *passing type of* t *at* s. We also call $t(|s|)$ the *passing type of* t *at* c_n, where $n+1 = |s|$, as $|c_n| = n + 1$.

Let A be a subtree of $\mathbb{S}$, t be a node in $\mathbb{S}$, and c_n be a coding node in $\mathbb{S}$ such that $|c_n| < |t|$. We write $t(c_n; A)$ to denote the set of those formulas in $t(|c_n|)$ in which all parameters are from among $\{v_i : i \in \mathrm{N}^A_m \cup \{n\}\}$, where m is least such that $|c^A_m| \geq |c_n|$. We call $t(c_n; A)$ the *passing type of* t *at* c_n *over* A.

Given a coding node c^A_n in A, we write $t(n; A)$ to denote $t(c^A_n; A)$, and call this the *passing type of* t *at* n *over* A.

Note that passing types are partial types which do not include any unary relation symbols. Thus, one can have realizations of the same passing type by elements which differ on the unary relations. Further, note that the passing type of t at s only takes into consideration the length of s, not s itself. Writing the "passing type of t at s" rather than "passing type of t at $|s|$" continues the convention set forth in [42], [30], and continued in all papers following on these two.

Remark 4.12. In the case where the language $\mathcal{L}$ only has binary relation symbols, passing type reduces to the concept of passing number, first defined and used in [42] and [30] and later used in [13], [11], [10], [47]. This is because for binary relational structures, the tree $\mathbb{S}$ has a bounded degree of branching. In the special case of the Rado graph, where the language has exactly one binary relation, say E, the tree $\mathbb{S}$ is regular 2-branching and may be correlated with the tree of finite sequences of 0's and 1's; then the passing number 0 of t at s corresponds to the passing type generated by $\{\neg R(x, v_{|s|})\}$, and the passing number 1 of t at s corresponds to the passing type generated by $\{R(x, v_{|s|})\}$.

In the case of the rationals, the coding tree of 1-types $\mathbb{S}$ for $\mathbb{Q}$ provides a minimalistic way to view the work of Devlin in [9], as $\mathbb{S}$ branches exactly at coding nodes and nowhere else. In our set-up, any antichain of coding nodes is automatically a so-called diagonal antichain, as defined in Subsection 4.3. This differs from the previous approaches to big Ramsey degrees of $\mathbb{Q}$ in [32] and [9] (see also [45]), which use the binary branching tree, Milliken's theorem, and the method of envelopes.

We will need to be able to compare structures represented by different sets of coding nodes in $\mathbb{S}$. The next notion provides a way to do so.

Recall that x is the variable used in all 1-types in $\mathbb{S}$. Given subsets X and Y of ω and map $f : X \to Y$, let $f^* : X \cup \{x\} \to Y \cup \{x\}$ be the extension of f given by $f^*(x) = x$.

Definition 4.13 (Similarity of Passing Types over Subsets)**.** Let A and B be subsets of $\mathbb{S}$, and let $m, n \in \omega$ be such that $\mathrm{N}^A \cap m$ has the same number of elements as $\mathrm{N}^B \cap n$, say p. Let f be the increasing bijection from N^A_p to N^B_p. Suppose $s, t \in \mathbb{S}$ are such that $|c_m| < |s|$ and $|c_n| < |t|$. We write

$$s(c_m; A) \sim t(c_n; B) \tag{10}$$

when, given any relation symbol $R \in \mathcal{L}$ of arity k and k-tuple $(z_0, \dots, z_{k-1})$, where all z_i are from among $\{v_i : i \in \mathrm{N}^A_p\} \cup \{x\}$ and at least one z_i is the variable x, we have that $R(z_0, \dots, z_{k-1})$ is in $s(c_m; A)$ if and only if $R(f^*(z_0), \dots, f^*(z_{k-1}))$ is in $t(c_n; B)$. When $s(c_m; A) \sim t(c_n; B)$ holds, we say that the passing type of s at c_m over A is *similar* to the passing type of t at c_n over B.

If A and B each have at least $n+1$ coding nodes, then for $s,t \in \mathbb{S}$ with $|c^A_n| < |s|$ and $|c^B_n| < |t|$, define

$$s(n;A) \sim t(n;B) \tag{11}$$

to mean that $s(c^A_n;A) \sim t(c^B_n;B)$. When $s(n;A) \sim t(n;B)$, we say that *s over A and t over B have similar passing types at the n-th coding node*, or that *the passing type of s at n over A is similar to the passing type of t at n over B*.

It is clear that for fixed n, $\sim$ is an equivalence relation on passing types over subsets of $\mathbb{S}$.

The following fact is the essence of why we are interested in similarity of passing types: They tell us exactly when two structures represented by coding nodes are isomorphic as substructures of the enumerated structure $\mathbf{K}$; that is, when there exists an $\mathcal{L}$-isomorphism between the structures that preserves the order relation on their underlying sets inherited from ω.

Fact 4.14. *Let A and B be subsets of $\mathbb{S}$ and $n < \omega$ such that A and B each have $n+1$ many coding nodes. Then the substructures $\mathbf{K}\restriction A$ and $\mathbf{K}\restriction B$ are isomorphic, as ordered substructures of $\mathbf{K}$, if and only if*

1. *For each $i \le n$, the 1-types c^A_i and c^B_i contain the same parameter-free formulas; and*
2. *For all $i < j \le n$, $c^A_j(i;A) \sim c^B_j(i;B)$.*

We now extend the similarity relation on passing types over subsets of $\mathbb{S}$ to a relation on subtrees of $\mathbb{S}$ that preserves tree structure. For this, we first define a (strict) linear order $\prec$ on $\mathbb{S}$: We may assume there is a linear ordering on the relation symbols and negated relation symbols in $\mathcal{L}$, with the convention that all the negated relation symbols appear in the linear order before the relation symbols. (We make this convention to support the intuition that "moving left" from a node in a tree indicates that a relation does not hold, while "moving right" suggests that it does; the convention is not necessary for our results.) Extend the usual linear order $<$ on ω, the underlying set of $\mathbf{K}$, to the set $\{x\} \cup \omega$ by setting $x < n$ for each $n \in \omega$. Let $(\{x\} \cup \omega)^{<\omega}$, the set of finite sequences from $\{x\} \cup \omega$, have the induced lexicographic order. Then the induced lexicographic order on the set

$$(\{R : R \in \mathcal{L}\} \cup \{\neg R : R \in \mathcal{L}\}) \times (\{x\} \cup \omega)^{<\omega}$$

is a linear order on the set of atomic and negated atomic formulas of $\mathcal{L}$ that have one free variable x and parameters from ω. Since any node of $\mathbb{S}$ is completely determined by such atomic and negated atomic formulas, this lexicographic order gives rise to a linear order on $\mathbb{S}$, which we denote $\prec$. Observe that by definition of the lexicographic ordering, we have: If $s \subsetneq t$, then $s \prec t$; and for any incomparable $s,t \in \mathbb{S}$, if $|s \wedge t| = n$, then $s \prec t$ if and only if $s\restriction(n+1) \prec t\restriction(n+1)$. This order $\prec$ generalizes the lexicographic order for the case of binary relational structures in [42], [30], [13], [11], and [47].

Definition 4.15 (Similarity Map)**.** Let S and T be meet-closed subsets of $\mathbb{S}$. A function $f : S \to T$ is a *similarity map* of S to T if for all nodes $s,t \in \mathbb{S}$, the following hold:

1. f is a bijection which preserves $\prec$: $s \prec t$ if and only if $f(s) \prec f(t)$.
2. f preserves meets, and hence splitting nodes: $f(s \wedge t) = f(s) \wedge f(t)$.
3. f preserves relative lengths: $|s| < |t|$ if and only if $|f(s)| < |f(t)|$.

(4) f preserves initial segments: $s \subseteq t$ if and only if $f(s) \subseteq f(t)$.
(5) f preserves coding nodes and their parameter-free formulas: Given a coding node $c^S_n \in S$, $f(c^S_n) = c^T_n$; moreover, for $\gamma \in \Gamma$, $\gamma(v^S_n)$ holds in $\mathbf{K}$ if and only if $\gamma(v^T_n)$ holds in $\mathbf{K}$, where v^S_n and v^T_n are the vertices of $\mathbf{K}$ represented by coding nodes c^S_n and c^T_n, respectively.
(6) f *preserves relative passing types* at coding nodes: $s(n; S) \sim f(s)(n; T)$, for each n such that $|c^S_n| < |s|$.

When there is a similarity map between S and T, we say that S and T are *similar* and we write $S \sim T$. Given a subtree S of $\mathbb{S}$, we let $\mathrm{Sim}(S)$ denote the collection of all subtrees $T$of $\mathbb{S}$ which are similar to S. If $T' \subseteq T$ and f is a similarity map of S to T', then we say that f is a *similarity embedding* of S into T.

Remark 4.16. It follows from (2) that s is a splitting node in S if and only if $f(s)$ is a splitting node in T. Moreover, if s is a splitting node in S, then s has the same number of immediate successors in S as $f(s)$ has in T. Similarity is an equivalence relation on the subtrees of $\mathbb{S}$, since the identity map is a similarity map, the inverse of a similarity map is a similarity map, and the composition of two similarity maps is a similarity map.

Our notion of *similarity* extends the notion of *strong similarity* in [42] and [30] for trees without coding nodes, and in [13] and [11] for trees with coding nodes. We drop the word *strong* to make the terminology more efficient, since there is only one notion of similarity used in this paper.

Given two substructures $\mathbf{F}, \mathbf{G}$ of $\mathbf{K}$, we write $\mathbf{F} \cong^\omega \mathbf{G}$ when there exists an $\mathcal{L}$-isomorphism between $\mathbf{F}$ and $\mathbf{G}$ that preserves the linear order on their universes inherited from ω. Note that for any subtrees S, T of $\mathbb{S}$, $S \sim T$ implies that $\mathbf{K} \restriction S \cong^\omega \mathbf{K} \restriction T$.

4.3. **Diagonal coding trees and SDAP$^+$.** There are essentially four ways of approaching big Ramsey degrees of structures with relations of arity higher than two. The naïve approach is to fix an enumerated Fraïssé structure and use Milliken's theorem directly on a tree similar to $\mathbb{S}$ but without coding nodes – the traditional approach which works for unconstrained binary relational structures – and follow the proof outline in [42]. This approach fails outright; the standard method of using envelopes fails to provide upper bounds because there will be more than one (in fact, unboundedly many) similarity types of a given finite antichain inside a given finite subtree of the ambient tree.

A purely combinatorial approach has been developed in [2] and [3] for the 3-uniform hypergraph, by using the product Milliken theorem with auxiliary matrices to keep track of coding substructure.

A third possible approach would be to work with coding trees and use envelopes after first proving a Milliken-style theorem of the following somewhat different form: Given a finite coloring of all copies of a finite initial segment A of the coding tree $\mathbb{S}$, there is a subtree S of $\mathbb{S}$ which is similar to $\mathbb{S}$ such that all copies of A in S have the same color. It is conceivable that this statement might be true in our setting; the issue is that one would need to develop a new Ramsey theorem for products of sets of 1-types. Even if this statement is true, it still would leave one with the task of developing the right notion of envelope for each situation to prove upper bounds for the big Ramsey degrees.

Our approach starts with the kinds of trees that will actually produce exact big Ramsey degrees, upon taking a subcopy of $\mathbf{K}$ represented by an antichain of coding nodes in such trees. Namely, we will work with *diagonal coding trees*. This will enable us to characterize the big Ramsey degrees without the need to develop a notion of envelope and pass through that intermediate step. As this can be done with very little additional work in the forcing arguments, this is our approach: We will work with skew subtrees of $\mathbb{S}$ which have two-branching from the outset. This leads to the first direct proof of exact big Ramsey degrees, without any appeal to envelopes.

The following modification of Definition 4.1 of $\mathbb{S}(\mathbf{K})$ will be useful especially for Fraïssé classes which have both non-trivial unary relations and a linear order or some similar relation, such as the betweenness relation. Recall that Γ denotes the set of complete 1-types having only parameter-free formulas; in particular, the only relation symbols that can occur in any $\gamma \in \Gamma$ will be unary.

Definition 4.17 (The Unary-Colored Coding Tree of 1-Types, $\mathbb{U}(\mathbf{K})$)**.** Let $\mathcal{K}$ be a Fraïssé class in language $\mathcal{L}$ and $\mathbf{K}$ an enumerated Fraïssé structure for $\mathcal{K}$. For $n < \omega$, let c_n denote the 1-type of v_n over $\mathbf{K}_n$ (exactly as in the definition of $\mathbb{S}(\mathbf{K})$). Let $\mathcal{L}^-$ denote the collection of all relation symbols in $\mathcal{L}$ of arity greater than one, and let $\mathbf{K}^-$ denote the reduct of $\mathbf{K}$ to $\mathcal{L}^-$ and $\mathbf{K}_n^-$ the reduct of $\mathbf{K}_n$ to $\mathcal{L}^-$.

For $n < \omega$, define the *n-th level,* $\mathbb{U}(n)$, to be the collection of all 1-types s over $\mathbf{K}_n^-$ in the language $\mathcal{L}^-$ such that for some $i \geq n$, v_i satisfies s. Define $\mathbb{U}$ to be $\bigcup_{n<\omega} \mathbb{U}(n)$. The tree-ordering on $\mathbb{U}$ is simply inclusion. The *unary-colored coding tree of* 1*-types* is the tree $\mathbb{U}$ along with the function $c : \omega \to \mathbb{U}$ such that $c(n) = c_n$. Thus, c_n is the 1-type (in the language $\mathcal{L}^-$) of v_n in $\mathbb{U}(n)$ along with the additional "unary color" $\gamma \in \Gamma$ such that $\gamma(v_n)$ holds in $\mathbf{K}$.

Note that $\mathbf{K}^-$ is not necessarily a Fraïssé structure, as the collection of reducts of members of $\mathcal{K}$ to $\mathcal{L}^-$ need not be a Fraïssé class. This poses no problem to our uses of $\mathbb{U}$ or to the results.

Remark 4.18. In the case that $\mathcal{K}$ has no unary relations, $\mathbb{U}$ is the same as $\mathbb{S}$. Otherwise, the difference between $\mathbb{U}$ and $\mathbb{S}$ is that all non-coding nodes in $\mathbb{U}$ are complete 1-types over initial segments of $\mathbf{K}^-$ in the language $\mathcal{L}^-$, while all nodes in $\mathbb{S}$, coding or non-coding, are complete 1-types over initial segments of $\mathbf{K}$ in the language $\mathcal{L}$. In particular, $\mathbb{S}(0)$ equals Γ, while $\mathbb{U}(0)$ has exactly one node, c_0.

Definition 4.11 of passing type applies to $\mathbb{U}$, as the notion of passing type involves no unary relations. Definition 4.13 of similarity of passing types and Definition 4.15 of similarity maps both apply to $\mathbb{U}$, since the notion of coding nodes is the same in both $\mathbb{S}$ and $\mathbb{U}$. Working inside $\mathbb{U}$ instead of $\mathbb{S}$ makes the upper bound arguments for Fraïssé classes with both a linear order and unary relations simpler, lining up with the previous approach for big Ramsey degrees of $\mathbb{Q}_n$ in [29]. This set-up will allow us to do one uniform forcing proof in the next section for all classes satisfying SDAP$^+$. For classes with SFAP, the exact bound proofs will return to the $\mathbb{S}$ setting.

Lastly, we point out that the tree $\mathbb{U}$ extends the approach used by Zucker in [47] for certain free amalgamation classes with binary and unary relations.

The following definition of *diagonal* is motivated by Definition 3.2 in [30], though our use of trees with coding nodes mandates a modified approach.

Definition 4.19 (Diagonal tree)**.** We call a subtree $T \subseteq \mathbb{S}$ or $T \subseteq \mathbb{U}$ *diagonal* if each level of T has at most one splitting node, each splitting node in T has degree two (exactly two immediate successors), and coding node levels in T have no splitting nodes.

In most currently known cases with finite big Ramsey degrees, any *persistent* similarity type, in the sense of Definition 2.4, is diagonal (though additional requirements are necessary for constrained free amalgamation classes as in [11], [13], and [47]). This idea is seen for unconstrained structures with finitely many binary relations in [30], where Laflamme, Sauer, and Vuksanovic prove, using Milliken's theorem and envelopes, that the persistent similarity types are diagonal. We will show below that almost all the structures considered in Section 3 have diagonal coding trees.

Notation 4.20. Given a diagonal subtree T (of $\mathbb{S}$ or $\mathbb{U}$) with coding nodes, we let $\langle c_n^T : n < N\rangle$, where $N \le \omega$, denote the enumeration of the coding nodes in T in order of increasing length. Let ℓ_n^T denote $|c_n^T|$, the *length* of c_n^T. We shall call a node in T a *critical node* if it is either a splitting node or a coding node in T. Let

$$\widehat{T} = \{t\restriction n : t \in T \text{ and } n \le |t|\}. \tag{12}$$

Given $s \in T$ that is not a splitting node in T, we let s^+ denote the immediate successor of s in $\widehat{T}$. Given any ℓ, we let $T\restriction \ell$ denote the set of those nodes in $\widehat{T}$ with length ℓ, and we let $T\downarrow\ell$ denote the union of the set of nodes in T of length less than ℓ with the set $T\restriction\ell$.

Extending Notation 4.10 to subtrees T of either $\mathbb{S}$ or $\mathbb{U}$, we write $\mathbf{K}\restriction T$ to denote the substructure of $\mathbf{K}$ on N^T, the set of vertices of $\mathbf{K}$ represented by the coding nodes in T.

Definition 4.21 (Diagonal Coding Subtree)**.** A subtree $T \subseteq \mathbb{U}$ is called a *diagonal coding subtree* if T is diagonal and satisfies the following properties:

1. $\mathbf{K}\restriction T \cong \mathbf{K}$.
2. For each $n < \omega$, the collection of 1-types in $T\restriction(\ell_n^T + 1)$ over $\mathbf{K}\restriction(T\downarrow\ell_n^T)$ is in one-to-one correspondence with the collection of 1-types in $\mathbb{U}(n+1)$.
3. Given $m < n$ and letting $A := T\downarrow(\ell_m^T - 1)$, if $c_n^T \supseteq c_m^T$ and if there is an $s \in \mathbb{U}(n+1)$ such that $s \supseteq c_n$ and $s(c_n; \mathbb{U}\downarrow(|c_m| - 1)) \sim (c_m^T)^+(c_m^T; A)$, then
$$(c_n^T)^+(c_n^T; A) \sim (c_m^T)^+(c_m^T; A).$$

Likewise, a subtree $T \subseteq \mathbb{S}$ is a *diagonal coding subtree* if the above hold with $\mathbb{U}$ replaced by $\mathbb{S}$.

Remark 4.22. Requirement (3) aids in the proofs in the next section and can be met by the Fraïssé limit of any Fraïssé class satisfying SDAP. Note that if $T \subseteq \mathbb{U}$ (or $T \subseteq \mathbb{S}$) satisfies (3), then any subtree S of T satisfying $S \sim T$ automatically satisfies (3).

Now we are prepared to define the Diagonal Coding Tree Property, which is an assumption in Definition 2.15 of SDAP$^+$. We say that a tree T is *perfect* if T has no terminal nodes, and each node in T has at least two incomparable extensions in T.

Recall our assumption that any Fraïssé class $\mathcal{K}$ that we consider has at least one non-unary relation symbol in its language. We make this assumption because if $\mathcal{K}$ has only unary relation symbols in its language, then $\mathbb{S}$ is a disjoint union of finitely many infinite branches. In this case, finitely many applications of Ramsey's Theorem will yield finite big Ramsey degrees.

We point out that whenever $\mathcal{K}$ satisfies SFAP, every node in $\mathbb{S}$ (and also in $\mathbb{U}$) has at least two immediate successors. However, there are Fraïssé classes in binary relational languages that satisfy SDAP, and yet for which the trees $\mathbb{S}$ and $\mathbb{U}$ are not perfect; for example, certain Fraïssé classes of ultrametric spaces. In such cases, Theorem 5.12 does not apply, as the forcing posets used in its proof are atomic. Thus, one of the requirements for SDAP$^+$ is that there is a perfect subtree of $\mathbb{U}$ which codes a copy of $\mathbf{K}$, whenever $\mathcal{L}$ has relation symbols of arity greater than one. This is an ingredient in the next property.

Definition 4.23 (Diagonal Coding Tree Property)**.** A Fraïssé class $\mathcal{K}$ in language $\mathcal{L}$ satisfies the *Diagonal Coding Tree Property* if given any enumerated Fraïssé structure $\mathbf{K}$ for $\mathcal{K}$, there is a diagonal coding subtree T of either $\mathbb{S}$ or $\mathbb{U}$ such that T is perfect.

From here through most of Section 5, we will simply work in $\mathbb{U}$ to avoid duplicating arguments, noting that for Fraïssé classes with SFAP, or without SFAP but with Fraïssé limits having SDAP$^+$ and in a language with no unary relation symbols, the following can all be done inside $\mathbb{S}$.

We now define the space of coding subtrees of $\mathbb{U}$ with which we shall be working.

Definition 4.24 (The Space of Diagonal Coding Trees of 1-Types, $\mathcal{T}$)**.** Let $\mathbf{K}$ be any enumerated Fraïssé structure and let $\mathbb{T}$ be a fixed diagonal coding subtree of $\mathbb{U}$. Then the space of coding trees $\mathcal{T}(\mathbb{T})$ consists of all subtrees T of $\mathbb{T}$ such that $T \sim \mathbb{T}$. Members of $\mathcal{T}(\mathbb{T})$ are called simply *coding trees*, where diagonal is understood to be implied. We shall usually simply write $\mathcal{T}$ when $\mathbb{T}$ is clear from context. For $T \in \mathcal{T}$, we write $S \le T$ to mean that S is a subtree of T and S is a member of $\mathcal{T}$.

Remark 4.25. Given $\mathbb{T}$ satisfying (1)–(3) in Definition 4.21, if $T \subseteq \mathbb{T}$ satisfies $T \sim \mathbb{T}$, then T also satisfies (1)–(3). Any tree T satisfying (1) and (2) has no terminal nodes and has coding nodes dense in T. Condition (2) implies that the Fraïssé structure $\mathbf{J} := \mathbf{K} \restriction T$ represented by T has the following property: For any $i - 1 < j < k$ in J satisfying $\mathbf{J} \restriction (i \cup \{j\}) \cong \mathbf{J} \restriction (i \cup \{k\})$, it holds that $\mathrm{tp}(j/\mathbf{K}_i) = \mathrm{tp}(k/\mathbf{K}_i)$; equivalently, that whenever two vertices in J are in the same orbit over $\mathbf{J}_i$ in $\mathbf{J}$, they are in the same orbit over $\mathbf{K}_i$ in $\mathbf{K}$.

The first use of diagonal subtrees of the infinite binary tree in characterizing exact big Ramsey degrees was for the rationals in [9]. Diagonal subtrees of the infinite binary tree turned out to be at the heart of characterizing the exact big Ramsey degrees of the Rado graph as well as of the generic directed graph and the generic tournament in [42] and [30]. More generally, diagonal subtrees of boundedly branching trees turned out to be central to the characterization of big Ramsey degrees of unconstrained structures with finitely many binary relations in [42] and [30]. More recently, characterizations of the big Ramsey degrees for triangle-free graphs were found to involve diagonal subtrees ([13],[12]), and similarly, for free amalgamation classes with finitely many binary relations and finitely many finite forbidden irreducible substructures on three or more vertices ([1],[11],[47]). However, in these cases, properties additional to being diagonal are essential to characterizing their big Ramsey degrees; hence, their big Ramsey degrees do not have a "simple" characterization solely in terms of similarity types of antichains of coding nodes in diagonal coding trees. We will prove that, similarly to the rationals and the Rado graph, all Fraïssé classes with Fraïssé structure satisfying SDAP$^+$ have big Ramsey degrees which are characterized simply by similarity types of antichains of coding nodes in diagonal coding trees, along with the passing types of their coding nodes.

Recalling from Notation 4.20 that $t \in T$ is called a *critical node* if t is either a splitting node or a coding node in T, any two critical nodes in a diagonal coding tree have different lengths, and thus, the levels of T are designated by the lengths of the critical nodes in T. (This follows from the definition of *diagonal*.) If $\langle d^T_m : m < \omega \rangle$ enumerates the critical nodes in T in order of strictly increasing length, then we let $T(m)$ denote the collection of those nodes in T with length $|d^T_m|$, which we call the *m-th level* of T.

Given a substructure $\mathbf{J}$ of $\mathbf{K}$, we let $\mathbb{U} \restriction \mathbf{J}$ denote the subtree of $\mathbb{U}$ induced by the meet-closure of the coding nodes $\{c_n : n \in \mathrm{J}\}$. We call $\mathbb{U} \restriction \mathbf{J}$ the *subtree of* $\mathbb{U}$ *induced by* $\mathbf{J}$. If $\mathbf{J} = \mathbf{K} \restriction T$ for some $T \in \mathcal{T}$, then $\mathbb{U} \restriction \mathbf{J} = T$, as T being diagonal ensures that the coding nodes in $\mathbb{U} \restriction \mathbf{J}$ are exactly those in T.

We now state the property truly at the heart of this paper. This is the property which makes the forcing arguments in the next section simpler than the arguments for binary relational structures omitting some finite set of irreducible substructures on three or more vertices. As seen in Section 3, this property unifies a seemingly disparate collection of Fraïssé classes with finite big Ramsey degrees.

Definition 4.26 (SDAP$^+$, Coding Tree Version)**.** A Fraïssé class $\mathcal{K}$ satisfies the *Coding Tree Version of SDAP$^+$* if and only if $\mathcal{K}$ satisfies the disjoint amalgamation property and, letting $\mathbf{K}$ be any enumerated Fraïssé limit of $\mathcal{K}$, $\mathbf{K}$ satisfies the Diagonal Coding Tree Property, the Extension Property, and the following condition:

Let T be any diagonal coding subtree of $\mathbb{U}(\mathbf{K})$ (or of $\mathbb{S}(\mathbf{K})$), and let $\ell < \omega$ be given. Let i, j be any distinct integers such that $\ell < \min(|c^T_i|, |c^T_j|)$, and let $\mathbf{C}$ denote the substructure of $\mathbf{K}$ represented by the coding nodes in $T \downarrow \ell$ along with $\{c^T_i, c^T_j\}$. Then there are $m \geq \ell$ and $s', t' \in T \restriction m$ such that $s' \supseteq s$ and $t' \supseteq t$ and, assuming (1) and (2), the conclusion holds:

1. Suppose $n \geq m$ and $s'', t'' \in T \restriction n$ with $s'' \supseteq s'$ and $t'' \supseteq t'$.
2. Suppose $c^T_{i'} \in T$ is any coding node extending s''.

Then there is a coding node $c^T_{j'} \in T$, with $j' > i'$, such that $c_{j'} \supseteq t''$ and the substructure of $\mathbf{K}$ represented by the coding nodes in $T \downarrow \ell$ along with $\{c^T_{i'}, c^T_{j'}\}$ is isomorphic to $\mathbf{C}$.

Remark 4.27. The structures $\mathbf{K} \restriction (T \downarrow \ell)$, $\mathbf{K} \restriction (T \downarrow m)$, and $\mathbf{K} \restriction (T \downarrow n)$ above play the roles of $\mathbf{A}$, $\mathbf{A}'$, and $\mathbf{B}$, respectively, in Definition 2.10. The Extension Property will be defined in the next section, in Definition 5.5. Suffice it to say here that the Extension Property is easily satisfied by Fraïssé limits of all classes satisfying SFAP, and of all classes satisfying SDAP that are considered in this paper. It may even turn out to follow from SDAP, but we include it to be precise about what is being assumed in SDAP$^+$.

The final work in this subsection is to prove that most of the Fraïssé classes with SDAP discussed in Section 3 have Fraïssé limits satisfying the Diagonal Coding Tree Property, thus completing another part of the proofs that these classes have Fraïssé limits satisfying SDAP$^+$. Verification that these Fraïssé limits satisfy the Extension Property will be carried out in Section 5.

We begin with free amalgamation classes satisfying SFAP. For Fraïssé limits of these classes, we can always construct a diagonal coding subtree of $\mathbb{S}$. The following notation will be used in the rest of this subsection. Given $j < \omega$, sets vertices $\{v_{m_i} : i < j\}$ and $\{v_{n_i} : i < j\}$, and 1-types $s, t \in \mathbb{S}$ such that $|s| > m_{j-1}$ and $|t| > n_{j-1}$, we will write

$$s \restriction (\mathbf{K} \restriction \{v_{m_i} : i < j\}) \sim t \restriction (\mathbf{K} \restriction \{v_{n_i} : i < j\}) \tag{13}$$

exactly when, for each $i < j$, $s(c_{m_i}; \{c_{m_k} : k < i\}) \sim t(c_{n_i}; \{c_{n_k} : k < i\})$.

Theorem 4.28. *SFAP implies SDAP$^+$.*

Proof. Suppose $\mathcal{K}$ is a Fraïssé class satisfying SFAP. Then $\mathcal{K}$ automatically also satisfies SDAP.

Let $\mathbf{K}$ be any enumerated Fraïssé structure for $\mathcal{K}$, and let $\mathbb{S}$ be the coding tree of 1-types over finite initial segments of $\mathbf{K}$. Recall that c_n denotes the n-th coding node of $\mathbb{S}$, that is, the 1-type of the n-th vertex of $\mathbf{K}$ over $\mathbf{K}_n$. If there are any unary relations in the language $\mathcal{L}$ for $\mathcal{K}$, then $\mathbb{S}(0)$ will have more than one node. Recall our convention that the "leftmost" or $\prec$-least node in $\mathbb{S}(n)$ is the 1-type over $\mathbf{K}_n$ in which no relations of arity greater than one are satisfied.

We start constructing a diagonal coding subtree $\mathbb{T}$ by letting the minimal level of $\mathbb{T}$ equal $\mathbb{S}(0)$. Take a level set X of $\mathbb{S}$ satisfying (a) for each $t \in \mathbb{S}(0)$, the number of nodes in X extending t is the same as the number of nodes in $\mathbb{S}(1)$ extending t, and (b) the subtree U_0 generated by the meet-closure of X is diagonal. We may assume, for convenience, that the $\prec$-order of the splitting nodes in U_0 is the same as the ordering by their lengths.

Let x_* denote the $\prec$-least member of X extending c_0. (If there are no unary relation symbols in the language, then x_* is the "leftmost" or $\prec$-least node in X.) Let $c_0^{\mathbb{T}}$ denote the coding node of least length extending x_*. Extend the rest of the nodes in X to the length of $c_0^{\mathbb{T}}$ and call this set of nodes, along with $c_0^{\mathbb{T}}$, Y; define $\mathbb{T} \restriction |c_0^{\mathbb{T}}| = Y$. Then take one immediate successor in $\mathbb{S}$ of each member of Y so that there is a one-to-one correspondence between the 1-types in Y over $\mathbf{K} \restriction \{v_0^T\}$, where $v_0^{\mathbb{T}}$ is the vertex in $\mathbf{K}$ represented by $c_0^{\mathbb{T}}$, and the 1-types in $\mathbb{S}(1)$: Letting $p = |\mathbb{S}(1)|$, list the nodes in $\mathbb{S}(1)$ and Y in $\prec$-increasing order as $\langle s_i : i < p\rangle$ and $\langle y_i : i < p\rangle$, respectively. Take z_i to be an immediate successor of y_i in $\mathbb{S}$ such that $z_i \restriction (\mathbf{K} \restriction \{v_0^{\mathbb{T}}\}) \sim s_i$. Such z_i exist by SFAP. Let $\mathbb{T} \restriction (|c_0^{\mathbb{T}}| + 1|) = \{z_i : i < p\}$. This constructs $\mathbb{T}$ up to length $|c_0^{\mathbb{T}}| + 1$.

The rest of $\mathbb{T}$ is constructed similarly: Suppose $n \geq 1$ and $\mathbb{T}$ has been constructed up to the immediate successors of its $(n-1)$-st coding node, $c_{n-1}^{\mathbb{T}}$. Take W to be the set of nodes in $\mathbb{T}$ of length $|c_{n-1}^{\mathbb{T}}| + 1$. This set W has the same size as $\mathbb{S}(n)$; let $\varphi : W \to \mathbb{S}(n)$ be the $\prec$-preserving bijection. Take a level set X of nodes in $\mathbb{S}$ extending W so that (a) for each $w \in W$, the number of nodes in X extending w is the same as the number of nodes in $\mathbb{S}(n+1)$ extending $\varphi(w)$, and (b) the tree U generated by the meet-closure of X is diagonal. Again, we may assume that the splitting nodes in U increase in length as their $\prec$-order increases.

Note that X and $\mathbb{S}(n+1)$ have the same cardinality. Let $p = |\mathbb{S}(n+1)|$ and enumerate X in $\prec$-increasing order as $\langle x_i : i < p\rangle$. Let i_* be the index so that x_{i_*} is the $\prec$-least member of X extending $\varphi(c_n)$. Let $c_n^{\mathbb{T}}$ denote the coding node in $\mathbb{S}$ of shortest length extending x_{i_*}. For each $i \in p \setminus \{i_*\}$, take one $y_i \in \mathbb{S}$ of length $|c_n^{\mathbb{T}}|$ extending x_i. Let $y_{i_*} = c_n^{\mathbb{T}}$, $Y = \{y_i : i < p\}$, and $\mathbb{T} \restriction |c_n^{\mathbb{T}}| = Y$. Let $\langle s_i : i < p\rangle$ enumerate the nodes in $\mathbb{S}(n+1)$ in $\prec$-increasing order. Then for each $i < p$, let z_i

be an immediate successor of y_i in $\mathbb{S}$ satisfying

$$z_i \restriction (\mathbf{K} \restriction \{v_m^{\mathbb{T}} : m \le n\}) \sim s_i \restriction \mathbf{K}_{n+1}, \tag{14}$$

where $v_m^{\mathbb{T}}$ is the vertex of $\mathbf{K}$ represented by the coding node $c_m^{\mathbb{T}}$. Again, such z_i exist by SFAP. Let $\mathbb{T} \restriction (|c_n^{\mathbb{T}}| + 1) = \{z_i : i < p\}$.

In this manner, we construct a subtree $\mathbb{T}$ of $\mathbb{S}$. It is straightforward to check that this construction satisfies (1) and (2) of Definition 4.21 of diagonal coding tree. Using SFAP, we may construct $\mathbb{T}$ so that property (3) holds. As long as the language for $\mathcal{K}$ contains at least one relation symbol, $\mathbb{T}$ will be a perfect tree. Thus, any Fraïssé limit for $\mathcal{K}$ satisfies the Diagonal Coding Tree Property. We show in Lemma 5.7 that any Fraïssé limit for $\mathcal{K}$ will also satisfy the Extension Property, completing the proof that Fraïssé limits of Fraïssé classes with SFAP have SDAP$^+$. □

Remark 4.29. The same argument as in Theorem 4.28 shows that the generic tournament has the Diagonal Coding Tree Property. In fact, the same argument works for all unconstrained binary relational structures (see [30]), because such structures have coding trees of 1-types which are everywhere 2^k-branching, where k is the number of binary relations. These are essentially the same constructions as the diagonal antichains constructed in [30] to characterize the exact big Ramsey degrees for such structures.

The next lemma completes the proofs of Propositions 3.7, 3.8, and 3.11.

Lemma 4.30. *There is a diagonal coding tree representing $\mathbb{Q}_n$, for each $n \ge 1$, and for the four non-trivial reducts of the rationals. Hence, these structures have the Diagonal Coding Tree Property.*

Proof. We have already seen in Figure 1 in Subsection 4.1 that $\mathbb{S}(\mathbb{Q}) = \mathbb{U}(\mathbb{Q})$ is a skew tree with binary splitting. Similarly, for $n \ge 2$, $\mathbb{U}(\mathbb{Q}_n)$ is a skew tree with binary splitting. In Figure 2, we have seen that $\mathbb{S}(\mathbb{Q}_n)$ consists of n-many trees which are isomorphic copies; $\mathbb{U}(\mathbb{Q}_n)$ consolidates these into one tree with n different "unary-colored" coding nodes. Thus, to construct a diagonal coding subtree $\mathbb{T}$ of $\mathbb{U}(\mathbb{Q}_n)$, it only remains to choose splitting nodes for $\mathbb{T}$ (which are coding nodes in $\mathbb{U}$ but not in $\mathbb{T}$) and then choose other coding nodes in $\mathbb{U}$ to be inherited as the coding nodes in $\mathbb{T}$, so as to satisfy requirements (2) and (3) of Definition 4.21, the definition of diagonal coding subtree. The construction is a slight modification of the one given in [29], where they constructed diagonal antichains of (non-coding) trees for $\mathbb{Q}_n$.

Take the only node in $\mathbb{U}(0)$, c_0, to be the least splitting node in $\mathbb{T}$. Let $\mathbb{T} \restriction 1$ consists of the two immediate successors of c_0 in $\mathbb{U}$, say $s_0 \prec s_1$. Then extend s_0 to the next coding node in $\mathbb{U}$, and label this node $c_0^{\mathbb{T}}$. If $n \ge 2$, we also require that $c_0^{\mathbb{T}}$ satisfies the same unary relations as c_0 does. Take any extension $t_1 \supseteq s_1$ in $\mathbb{U}$ of length $|c_0^{\mathbb{T}}|$. The set $\{t_0, t_1\}$ make up the nodes in $\mathbb{T}$ at the level of its least coding node, $c_0^{\mathbb{T}}$. Extend $c_0^{\mathbb{T}}$ $\prec$-leftmost in $\mathbb{U}$, call this node u_0. There is only one immediate successor of t_1 in $\mathbb{U}$, call it u_1. Let $\mathbb{T} \restriction (|c_0^{\mathbb{T}}| + 1) = \{u_0, u_1\}$.

In general, given $n \ge 1$ and $\mathbb{T}$ constructed up to nodes of length $|c_{n-1}^{\mathbb{T}}| + 1$, enumerate these nodes in $\prec$-increasing order as $\langle t_i : i < n + 2\rangle$. Let j denote the index of the node that will be extended to the next coding node, $c_n^{\mathbb{T}}$. This is the only node that needs to branch before the level of $c_n^{\mathbb{T}}$. Let s be the shortest splitting node in $\mathbb{U}$ extending t_j. Denote its immediate successors by s_0, s_1, where $s_0 \prec s_1$.

Let $c_n^{\mathbb{T}}$ be the coding node of least length in $\mathbb{S}$ extending s_0; if $n \geq 2$, also require that $c_n^{\mathbb{T}}$ satisfies the same unary relation as c_n. Extend all the nodes s_0 and t_i, $i \in (n+2) \setminus \{j\}$ to nodes in $\mathbb{S}$ of length $|c_n^{\mathbb{T}}|$. These nodes along with $c_n^{\mathbb{T}}$ construct $\mathbb{T} \restriction |c_n^{\mathbb{T}}|$. Take the $\prec$-leftmost extension of $c_n^{\mathbb{T}}$ to be its immediate successor in $\mathbb{T}$. All other nodes in $\mathbb{T} \restriction |c_n^{\mathbb{T}}|$ have only one immediate successor in $\mathbb{S}$, so there is no choice to be made.

This constructs a diagonal tree $\mathbb{T}$ representing a copy of $\mathbb{Q}_n$. Note that taking the $\prec$-leftmost extension of each coding node has the effect that all extensions of any coding node $c_n^{\mathbb{T}}$ in $\mathbb{T}$ include the formula $x < v_n^{\mathbb{T}}$, satisfying (3) of the definition of diagonal coding tree.

The construction of diagonal coding trees for the betweenness relation B, the ternary circular order relation K, and the quaternary separation relation S are achieved by the same process as for $\mathbb{Q}$, since (above the first two or three levels), their coding trees of 1-types are exactly like $\mathbb{S}(\mathbb{Q})$: only coding nodes split, and the splitting degree is two. □

Next, we consider ordered SFAP classes.

Lemma 4.31. *Suppose $\mathcal{K}$ is a Fraïssé class satisfying SFAP and let $\mathcal{K}^<$ denote the Fraïssé class of ordered expansions of members of $\mathcal{K}$. Then the Fraïssé limit $\mathbf{K}^<$ of $\mathcal{K}^<$ satisfies SDAP$^+$.*

Proof. Let $\mathcal{L}$ denote the language for $\mathcal{K}$, and let $\mathcal{L}^*$ be the expansion $\mathcal{L} \cup \{<\}$, the language of $\mathcal{K}^<$. Let $\mathbf{K}^<$ denote an enumerated structure for $\mathcal{K}^<$, and let $\mathbf{K}$ denote the reduct of $\mathbf{K}^<$ to $\mathcal{L}$; thus, $\mathbf{K}$ is an enumerated Fraïssé structure for $\mathcal{K}$. The universes of $\mathbf{K}$ and $\mathbf{K}^<$ are ω, which we shall denote as $\langle v_n : n < \omega \rangle$. Let $\mathbb{U}$ denote the coding tree of 1-types induced by $\mathbf{K}$, and $\mathbb{U}^<$ denote the coding tree of 1-types induced by $\mathbf{K}^<$. As usual, we let c_n denote the n-th coding node in $\mathbf{K}$, and we will let $c_n^<$ denote the n-th coding node in $\mathbf{K}^<$. (Normally, if $<$ is in the language of a Fraïssé class $\mathcal{K}$, then we will simply write $\mathbf{K}$ for its enumerated Fraïssé structure and $\mathbb{U}$ for its induced coding tree of 1-types, but here it will aid the reader to consider the juxtaposition of $\mathbb{U}$ and $\mathbb{U}^<$.) Notice that $\mathcal{K}^<$ satisfies SDAP: This holds because SFAP implies SDAP, $\mathcal{LO}$ satisfies SDAP, and SDAP is preserved under free superposition. So it only remains to show that there is a diagonal coding tree for $\mathbf{K}^<$.

Note that since $\mathcal{L}$ has at least one non-unary relation symbol and since $\mathcal{K}$ satisfies SFAP, every node in the tree $\mathbb{U}$ has at least two immediate successors. The branching of $\mathbb{U}$ and $\mathbb{U}^<$ are related in the following way: Each node $t \in \mathbb{U}(0)^<$ has twice as many immediate successors in $\mathbb{U}(1)^<$ as its reduct to $\mathcal{L}$ has in $\mathbb{U}(1)$. In general, for $n \geq 1$, given a node $t \in \mathbb{U}^<(n)$, let s denote the collection of formulas in t using only relation symbols in $\mathcal{L}$ and note that $s \in \mathbb{U}(n)$. The number of immediate successors of t in $\mathbb{U}^<(n+1)$ is related to the number of immediate successors of s in $\mathbb{U}(n+1)$ as follows: Let $(*)_n(t)$ denote the following property:

$$(*)_n(t)\text{: } \{m < n : (x < v_m) \in t\} = \{m < n : (x < v_m) \in c_n^<\}$$

If $(*)_n(t)$ holds, then t has twice as many immediate successors in $\mathbb{U}^<(n+1)$ as s has in $\mathbb{U}(n)$, owing to the fact that each 1-type in $\mathbb{U}(n+1)$ extending s can be augmented by either of $(x < v_n)$ or $(v_n < x)$ to form an extension of t in $\mathbb{U}^<(n+1)$. If $(*)_n(t)$ does not hold, then any vertex v_i, $i > n$, satisfying t lies in an interval of the $<$-linearly ordered set $\{v_m : m < n\}$, where neither of the endpoints are v_n.

Thus, the order between v_i and v_n is already determined by t; hence t has the same number of immediate successors in $\mathbb{U}^<(n+1)$ as s has in $\mathbb{U}(n)$.

A diagonal coding subtree $\mathbb{T}^<$ of $\mathbb{U}^<$ can be constructed similarly as in Theorem 4.28 with the following modifications: Suppose $\mathbb{T}^<$ has been constructed up to a level set W, where either $n = 0$ and $W = \mathbb{U}^<(0) := \{c_0^<\}$, or else $n \geq 1$ and W is the set of immediate successors of the $(n-1)$-st coding node of $\mathbb{T}^<$. This set W has the same size as $\mathbb{U}^<(n)$; let $\varphi : W \to \mathbb{U}^<(n)$ be the $\prec$-preserving bijection. As $\mathcal{K}$ is a free amalgamation class, we may assume that for any $s \in \mathbb{U}^<(n)$, if t, u in $\mathbb{U}^<(n+1)$ are immediate successors of s with $(x < v_n) \in t$ and $(v_n < x) \in u$, then $t \prec u$. Note that the two $\prec$-least extensions of s either both contain $(x < v_n)$, or else both contain $(v_n < x)$. Moreover, we may assume that the $\prec$-least immediate successor of s contains negations of all relations in s with v_n as a parameter.

Take a level set X of nodes in $\mathbb{U}^<$ extending W so that the following hold: (a) for each $w \in W$, the number of nodes in X extending w is the same as the number of nodes in $\mathbb{U}^<(n+1)$ extending $\varphi(w)$, and (b) the tree U generated by the meet-closure of X is diagonal, where each splitting node in U is extended by its two $\prec$-least immediate successors in $\mathbb{U}^<$, and all non-splitting nodes are extended by the $\prec$-least extension in $\mathbb{U}^<$. As in Theorem 4.28, we may assume that the splitting nodes in U increase in length as their $\prec$-order increases, though this has no bearing on the theorems in the next section.

Let $p := |\mathbb{U}^<(n+1)|$ and index the nodes in $\mathbb{U}^<(n+1)$ in $\prec$-increasing order as $\langle s_i : i < p\rangle$. Note that X has p-many nodes; index them in $\prec$-increasing order as $\langle x_i : i < p\rangle$. Let x_{i_*} denote the $\prec$-least member of X extending $\varphi(c_n^<)$, and extend x_{i_*} to a coding node in $\mathbb{U}^<$ satisfying the same $\gamma \in \Gamma$ as c_n; label it y_{i_*}. This node y_{i_*} will be the n-th coding node, $c_n^{\mathbb{T}^<}$, of the diagonal coding subtree $\mathbb{T}^<$ of $\mathbb{U}^<$ which we are constructing. For each $i \in p \setminus \{i_*\}$, take one $y_i \in \mathbb{U}^<$ of length $|c_n^{\mathbb{T}^<}|$ extending x_i so that y_i is the $\prec$-least extension of x_i, subject to the following: Let n_* be the index such that $c_n^{\mathbb{T}^<} = c_{n_*}^<$. For $i < p$, if $(v_n < x)$ is in s_i, then we take y_i so that for some $m < n_*$ such that $v_n < v_m$, $(v_m < x)$ is in y_i. This has the effect that if $(v_n < x)$ is in s_i, then any vertex v_j represented by a coding node extending s_i will satisfy $v_m < v_j$; and since $v_n < v_m$, it will follow that $v_n < v_j$; hence $(v_{n_*} < x)$ is automatically in y_i. Likewise, if $(x < v_n)$ is in s_i, then we take y_i so that for some $m < n_*$ such that $v_m < v_n$, $(x < v_m)$ is in y_i.

Let $Y = \{y_i : i < p\}$ and define the set of nodes in $\mathbb{T}^<$ at the level of $c_n^{\mathbb{T}^<}$ to be Y. For each $i < p$, let z_i be an immediate successor of y_i in $\mathbb{U}^<$ satisfying

$$z_i \restriction (\mathbf{K} \restriction \{v_j^{\mathbb{T}^<} : j \leq n\}) \sim s_i \restriction \mathbf{K}_{n+1}, \tag{15}$$

where $v_j^{\mathbb{T}^<}$ is the vertex represented by $c_j^{\mathbb{T}^<}$. This is possible by SFAP. For the linear order, this was taken care of by SDAP and our choice of y_i. Let $\mathbb{T}^< \restriction (|c_n^{\mathbb{T}}| + 1) = \{z_i : i < p\}$.

In this manner, we construct a coding subtree $\mathbb{T}^<$ of $\mathbb{U}^<$ which is diagonal, representing a substructure of $\mathbf{K}^<$ which is again isomorphic to $\mathbf{K}^<$. By extending coding nodes in $\mathbb{T}^<$ by their $\prec$-least extensions in $\mathbb{U}^< T$, we satisfy (3) of the definition of diagonal coding tree .

Hence, $\mathbf{K}^<$ satisfies the Diagonal Coding Tree Property, and the Extension Property trivially holds. Therefore, $\mathbf{K}^<$ satisfies SDAP$^+$. $\square$

Next, we construct diagonal coding trees for the linear order with a convexly ordered equivalence relation and its iterates of coarsenings, as well as for the "mixed" structures consisting of a single linear order, finitely many nested convexly ordered equivalence relations, and a vertex partition into finitely many definable dense pieces. The main difference between these structures and the ones considered in the previous lemmas, is that now one has to be careful when choosing splitting nodes, as not all splitting nodes can be extended to the same structures. The following lemma completes the proof of the Diagonal Coding Tree Property in Proposition 3.9.

Lemma 4.32. $(\mathbb{Q}_{\mathbb{Q}})_n$*, for each* $n \geq 1$*, has the Diagonal Coding Tree Property Moreover, the Fraïssé limit of any class* $\mathcal{K}$ *in* $\mathcal{LOE}_{1,n,p}$ *also has the Diagonal Coding Tree Property.*

Proof. We present the construction for $\mathbb{Q}_{\mathbb{Q}}$ and then discuss the construction for the more general case. Let $\mathbb{U}$ denote $\mathbb{U}(\mathbb{Q}_{\mathbb{Q}})$. It may aid the reader to recall Figure 3, where a graphic is presented for a particular enumeration of $\mathbb{Q}_{\mathbb{Q}}$.

We construct a subtree $\mathbb{T}$ of $\mathbb{U}$ which is diagonal and such that for each m, the immediate successors of the nodes in $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$ have 1-types over $\mathbb{Q}_{\mathbb{Q}}\restriction\{v^{\mathbb{T}}_j : j \leq m\}$ which are in one-to-one correspondence (in $\prec$-order) with the 1-types in $\mathbb{U}(m+1)$. The idea is relatively simple: We work our way from the outside (non-equivalence) inward (equivalence) in the way we construct the splitting nodes in $\mathbb{T}$.

Given $\mathbb{T}\restriction|c^{\mathbb{T}}_{m-1}|$, let $\varphi : \mathbb{U}(m) \to \mathbb{T}\restriction|c^{\mathbb{T}}_{m-1}|$ be the $\prec$-preserving bijection, and let t_* denote the node $\varphi(c_m)$ in $\mathbb{T}\restriction|c^{\mathbb{T}}_{m-1}|$. This t_* is the node which we need to extend to the next coding node. Recall that only the coding nodes in $\mathbb{U}$ have more than one immediate successor; so t_* is the only node we need to extend to one or three splitting nodes before making the level $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$.

The simplest case is when the coding node c_m has two immediate successors: these contain $\{x < v_m, xEv_m\}$ and $\{v_m < x, xEv_m\}$, respectively. First extend t_* to a coding node $c_i \in \mathbb{U}$, and then take extensions s_0, s_1 of this coding node so that $\{x < v_i, xEv_i\} \subseteq s_0$ and $\{v_i < x, xEv_i\} \subseteq s_1$. Extend s_0 to a coding node $c_j \in \mathbb{U}$, and define $c^{\mathbb{T}}_m = c_j$ and $v^{\mathbb{T}}_m = v_j$. Let u_0 be the extension of $c^{\mathbb{T}}_m$ in $\mathbb{U}$ which contains $\{x < v_j, xEv_j\}$. Extend s_1 to a node $t_1 \in \mathbb{U}\restriction|c^{\mathbb{T}}_m|$, and let u_1 be the immediate successor of t_1 in $\mathbb{U}$. Extend all other nodes in $\mathbb{T}\restriction|c^{\mathbb{T}}_{m-1}|$ (besides t_*) to a node in $\mathbb{U}$ of length $|c^{\mathbb{T}}_m|$, and let $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$ consist of these nodes along with t_0 and t_1. Let $\mathbb{T}\restriction(|c^{\mathbb{T}}_m|+1)$ consist of u_0, u_1, and one immediate successor of each of the nodes in $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$. By the transitivity of both relations $<$ and E, we obtain that the $\prec$-preserving bijection between $\mathbb{U}(m+1)$ and $\mathbb{T}\restriction(|c^{\mathbb{T}}_m|+1)$ preserves passing types over $\mathbb{Q}_{\mathbb{Q}}\restriction\{v^{\mathbb{T}}_k : k \leq m\}$.

If the coding node c_m has four immediate successors, then these extensions consist of all choices from among $\{x < v_m, v_m < x\}$ and $\{xEv_m, x\not\!\!Ev_m\}$. We start on the outside with non-equivalence and work our way inside to equivalence. First extend t_* to a coding node $c_i \in \mathbb{U}$ which has four immediate successors, and let s_0 denote the extension with $\{x < v_i, x\not\!\!Ev_i\}$ and s_3 denote the extension with $\{v_i < x, x\not\!\!Ev_i\}$. Again, extend s_0 to a coding node $c_j \in \mathbb{S}$ which has four immediate successors, and let s_0 denote the extension with $\{x < v_j, x\not\!\!Ev_j\}$ and s_1 denote the extension with $\{v_j < x, x\not\!\!Ev_j\}$. Then extend s_1 to any coding node c_k. Take c_ℓ to be a coding node extending $c_k \cup \{x < v_k, xEv_k\}$, and define $c^{\mathbb{T}}_m = c_\ell$ and $v^{\mathbb{T}}_m = v_\ell$. Let t_0 be the $\prec$-leftmost extension of s_0 in $\mathbb{U}(\ell)$, let t_2 be the $\prec$-leftmost extension of $c_k \cup \{v_k < x, xEv_k\}$ in $\mathbb{U}(\ell)$, and let t_3 be the $\prec$-leftmost extension of s_3 in $\mathbb{U}(\ell)$. Finally, define $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$ to consist of $\{t_0, c^{\mathbb{T}}_m, t_2, t_3\}$ along with the leftmost extensions in $\mathbb{U}(\ell)$ of the nodes in $(\mathbb{T}\restriction|c^{\mathbb{T}}_{m-1}|) \setminus \{t_*\}$. Let the nodes in $\mathbb{T}\restriction|c^{\mathbb{T}}_m + 1|$ consist of $c^{\mathbb{T}}_m \cup \{x < v^{\mathbb{T}}_m, xEv^{\mathbb{T}}_m\}$, along with the immediate successors in $\mathbb{U}(\ell+1)$ of the rest of the nodes in $\mathbb{T}\restriction|c^{\mathbb{T}}_m|$ It is routine to check that, by transitivity of the relations $<$ and E, the immediate successors

The idea for general $(\mathbb{Q}_{\mathbb{Q}})_n$ is similar. Here we have a sequence of convex equivalence relations $\langle E_i : i < n\rangle$, where for each $i < n-1$, E_{i+1} coarsens E_i. Similarly to the above, each coding node c_m has $2(j+1)$ many immediate successors, for some $j \le n$. The immediate successors run through all combinations of choices from among $\{x < v_m, v_m < x\}$ and $\{xE_0v_m\} \cup \{(xE_{i+1}v_m \wedge x\not{E}_iv_m) : i < j\}$. When constructing skew splitting, in order to set up so that the desired passing types are available at the next coding node of $\mathbb{T}$, we start on the "outside" with types containing $(xE_jv_m \wedge x\not{E}_{j-1}v_m)$ and work our way inward, with the increasingly finer equivalence relations, analogously to how the case of four immediate successors was handled above for $\mathbb{Q}_{\mathbb{Q}}$.

The presence of any unary relations has no effect on the existence of diagonal coding trees. □

5. Forcing exact upper bounds for big Ramsey degrees

This section contains the Ramsey theorem for colorings of copies of a given finite substructure of a Fraïssé structure satisfying SDAP$^+$. Theorem 5.22 provides upper bounds for the big Ramsey degrees of such structures, which turn out to be exact. The proof of exactness will be given in Section 6.

The key combinatorial content of Theorem 5.22 occurs in Theorem 5.12, where we use the technique of forcing to essentially conduct an unbounded search for a finite object, achieving within ZFC one color per level set extension of a given finite tree. It is important to note that we never actually go to a generic extension. In fact, the forced generic object is very much *not* a coding tree. Rather, we use the forcing to do two things: (1) Find a good set of nodes from which we can start to build a subtree which can have the desired homogeneity properties; and (2) Use the forcing to guarantee the existence of a finite object with certain properties. Once found, this object, being finite, must exist in the ground model.

We take here a sort of amalgamation of techniques developed in [13], [11], and [10], making adjustments as necessary. The main differences from previous work are the following: The forcing poset is on trees of 1-types; as such, we work with the general notion of passing type, in place of passing number used in the papers [10], [11], [13], and [47] for binary relational structures. Moreover, Definition 5.3 presents a stronger requirement than just similarity. This addresses both the fact that relations can be of any arity, and the fact that we consider Fraïssé classes which have disjoint, but not necessarily free, amalgamation.

We now set up notation, definitions, and assumptions for Theorem 5.12, beginning with the following convention: We also define the Extension Property, which is one of the conditions for SDAP$^+$ to hold.

Convention 5.1. Let $\mathcal{K}$ be a Fraïssé class in a language $\mathcal{L}$ and $\mathbf{K}$ a Fraïssé limit of $\mathcal{K}$. If (a) $\mathcal{K}$ satisfies SFAP, or (b) $\mathbf{K}$ satisfies SDAP$^+$ and either has no unary relations or has no transitive relations, then we work inside a diagonal coding subtree $\mathbb{T}$ of $\mathbb{S}$. Otherwise, we work inside a diagonal coding subtree $\mathbb{T}$ of $\mathbb{U}$.

Remark 5.2. All proofs in this section could be done working inside $\mathbb{U}$. Then in the case when $\mathcal{L}$ has unary relation symbols and $\mathbf{K}$ has no transitive relations, in order to obtain the optimal upper bounds, one would need to take a diagonal antichain of coding nodes, $\mathbb{D}$, in Lemma 5.20, which has the following properties: There exists a level ℓ such that $\mathbb{D}\restriction\ell$ has $|\Gamma|$-many nodes, labeled d_γ where $\gamma\in\Gamma$, such that for each coding node $c_n^{\mathbb{D}}$ in $\mathbb{D}$ there is exactly one $\gamma\in\Gamma$, call it γ^*, such that $c_n^{\mathbb{D}}$ extends d_{γ^*}. Further, for the vertex $v_n^{\mathbb{D}}$ represented by $c_n^{\mathbb{D}}$, $\gamma^*(v_n^{\mathbb{D}})$ holds in $\mathbf{K}$. The end result of this approach is equivalent to working in $\mathbb{S}$.

The results in this section could also be attained working solely in $\mathbb{S}$. However, in the case when $\mathcal{L}$ has unary relation symbols and $\mathbf{K}$ has a transitive relation, $\mathbb{S}$ will not contain a diagonal coding subtree, so the proofs would have to be modified to allow for more than one splitting node of a given length (see for instance, Figure 2 showing $\mathbb{S}(\mathbb{Q}_2)$). Convention 5.1 is intended to give the reader the idea of when each approach (working in $\mathbb{S}$ or working in $\mathbb{U}$) is most natural.

We point out that if $\mathcal{L}$ has no unary relation symbols, then $\mathbb{S}=\mathbb{U}$.

Let T be a diagonal coding tree for the Fraïssé limit $\mathbf{K}$ of some Fraïssé class $\mathcal{K}$. Recall that the tree ordering on T is simply inclusion. We recapitulate notation from Subsection 4.1: Each $t\in T$ can be thought of as a sequence $\langle t(i):i<|t|\rangle$ where $t(i)=(t\restriction\mathbf{K}_i)\setminus(t\restriction\mathbf{K}_{i-1})$. For $t\in T$ and $\ell\le|t|$, $t\restriction\ell$ denotes $\bigcup_{i<\ell}t(i)$, which we can think of as the sequence $\langle t(i):i<\ell\rangle$, the initial segment of t with domain ℓ. Note that $t\restriction\ell\in\mathbb{S}(\ell-1)$ (or $t\restriction\ell\in\mathbb{U}(\ell-1)$). (We let $\mathbb{S}(-1)=\mathbb{U}(-1)$ denote the set containing the empty set, just so that we do not have to always write $\ell\ge1$.)

The following extends Notation 4.20 to subsets of trees. For a finite subset $A\subseteq\mathbb{T}$, let

$$\ell_A=\max\{|t|:t\in A\}\quad\text{and}\quad\max(A)=\{s\in A:|s|=\ell_A\}. \tag{16}$$

For $\ell\le\ell_A$, let

$$A\restriction\ell=\{t\restriction\ell:t\in A\text{ and }|t|\ge\ell\} \tag{17}$$

and let

$$A{\downarrow}\ell=\{t\in A:|t|<\ell\}\cup A\restriction\ell. \tag{18}$$

Thus, $A\restriction\ell$ is a level set, while $A{\downarrow}\ell$ is the set of nodes in A with length less than ℓ along with the truncation to ℓ of the nodes in A of length at least ℓ. Notice that $A\restriction\ell=\emptyset$ for $\ell>\ell_A$, and $A{\downarrow}\ell=A$ for $\ell\ge\ell_A$. Given $A,B\subseteq T$, we say that B is an *initial segment* of A if $B=A{\downarrow}\ell$ for some ℓ equal to the length of some node in A. In this case, we also say that A *end-extends* (or just *extends*) B. If ℓ is not the length of any node in A, then $A{\downarrow}\ell$ is not a subset of A, but is a subset of $\widehat{A}$, where $\widehat{A}$ denotes $\{t\restriction n:t\in A\text{ and }n\le|t|\}$.

Define $\max(A)^+$ to be the set of nodes t in $T\restriction(\ell_A+1)$ such that t extends s for some $s\in\max(A)$. Given a node $t\in T$ at the level of a coding node in T, t has exactly one immediate successor in $\widehat{T}$, which we recall from Notation 4.20 is denoted as t^+.

Definition 5.3 (+-Similarity)**.** Let T be a diagonal coding tree for the Fraïssé limit $\mathbf{K}$ of a Fraïssé class $\mathcal{K}$, and suppose A and B are finite subtrees of T. We write $A\overset{+}{\sim}B$ and say that A and B are *+-similar* if and only if $A\sim B$ and one of the following two cases holds:

Case 1. If $\max(A)$ has a splitting node in T, then so does $\max(B)$, and the similarity map from A to B takes the splitting node in $\max(A)$ to the splitting node in $\max(B)$.

Case 2. If $\max(A)$ has a coding node, say c_n^A, and $f : A \to B$ is the similarity map, then $s^+(n; A) \sim f(s)^+(n; B)$ for each $s \in \max(A)$.

Note that $\stackrel{+}{\sim}$ is an equivalence relation, and $A \stackrel{+}{\sim} B$ implies $A \sim B$. When $A \sim B$ ($A \stackrel{+}{\sim} B$), we say that they have the same *similarity type* (*+-similarity type*).

Remark 5.4. For infinite trees S and T with no terminal nodes, $S \sim T$ implies that for each n, letting d_n^S and d_n^T denote the n-th critical nodes of S and T, respectively, $S \restriction |d_n^S| \stackrel{+}{\sim} T \restriction |d_n^T|$.

We adopt the following notation from topological Ramsey space theory (see [45]). Given $k < \omega$, we define $r_k(T)$ to be the restriction of T to the levels of the first k critical nodes of T; that is,

$$r_k(T) = \bigcup_{m<k} T(m), \tag{19}$$

where $T(m)$ denotes the set of all nodes in T with length equal to $|d_m^T|$. It follows from Remark 5.4 that for any $S, T \in \mathcal{T}$, $r_k(S) \stackrel{+}{\sim} r_k(T)$. Define $\mathcal{AT}_k$ to be the set of *k-th approximations* to members of $\mathcal{T}$; that is,

$$\mathcal{AT}_k = \{r_k(T) : T \in \mathcal{T}\}. \tag{20}$$

For $D \in \mathcal{AT}_k$ and $T \in \mathcal{T}$, define the set

$$[D, T] = \{S \in \mathcal{T} : r_k(S) = D \text{ and } S \leq T\}. \tag{21}$$

Lastly, given $T \in \mathcal{T}$, $D = r_k(T)$, and $n > k$, define

$$r_n[D, T] = \{r_n(S) : S \in [D, T]\}. \tag{22}$$

More generally, given any $A \subseteq T$, we use $r_k(A)$ to denote the first k levels of the tree induced by the meet-closure of A. We now have the necessary ideas to define the Extension Property.

Recall from Convention 5.1 that $\mathbb{T}$ is a fixed diagonal coding tree (in $\mathbb{S}$ or in $\mathbb{U}$) for an enumerated Fraïssé limit $\mathbf{K}$ of a Fraïssé class $\mathcal{K}$.

Definition 5.5 (Extension Property)**.** We say that $\mathbf{K}$ has the *Extension Property* when either (1) or (2) holds:

(1) Suppose A is a finite or infinite subtree of some $T \in \mathcal{T}$. Let k be given and suppose $\max(r_{k+1}(A))$ has a splitting node. Suppose that B is a +-similarity copy of $r_k(A)$ in T. Let u denote the splitting node in $\max(r_{k+1}(A))$, and let s denote the node in $\max(B)^+$ which must be extended to a splitting node in order to obtain a +-similarity copy of $r_{k+1}(A)$. If s^* is a splitting node in T extending s, then there are extensions of the rest of the nodes in $\max(B)^+$ to the same length as s^* resulting in a +-similarity copy of $r_{k+1}(A)$ which can be extended to a copy of A.

(2) There is some $2 \leq q < \omega$, and a function ψ defined on the set of splitting nodes in $\mathbb{T}$ and having range q, such that the following holds:
   (a) Suppose A is a finite or infinite subtree of some $T \in \mathcal{T}$. Let k be given and suppose $\max(r_{k+1}(A))$ has a splitting node. Suppose that B is a +-similarity copy of $r_k(A)$ in T such that the similarity map

$f : r_k(A) \to B$ has the property that for each splitting node $t \in r_k(A)$, $\psi(t) = \psi(f(t))$. Let u denote the splitting node in $\max(r_{k+1}(A))$, and let s denote the node in $\max(B)^+$ which must be extended to a splitting node in order to obtain a +-similarity copy of $r_{k+1}(A)$. Then for each $s' \supseteq s$ in T, there exists a splitting node $s^* \in T$ extending s' such that $\psi(s^*) = \psi(u)$. Moreover, given such an s^*, there are extensions of the rest of the nodes in $\max(B)^+$ to the same length as s^* resulting in a +-similarity copy of $r_{k+1}(A)$ which can be extended to a copy of A.

(b) The language for $\mathbf{K}$ has at least one binary relation symbol (besides equality), and the value of ψ is determined by some partition of all pairs of partial 1-types involving only binary relation symbols over a one-element structure into pieces $Q_0, \dots, Q_{q-1}$, such that whenever s is a splitting node in $\mathbb{T}$, $\psi(s) = m$ if and only if the following hold: whenever $c_j^{\mathbb{T}}, c_k^{\mathbb{T}}$ are coding nodes in $\mathbb{T}$ with $c_j^{\mathbb{T}} \wedge c_k^{\mathbb{T}} = s$, then the pair of partial 1-types of $v_j^{\mathbb{T}}$ and $v_k^{\mathbb{T}}$ over $\mathbf{K} \restriction \{v_i\}$ is in Q_m.

Remark 5.6. The Extension Property (1) easily holds for Fraïssé limits of all Fraïssé classes satisfying SFAP, as we show below in Lemma 5.7; and similarly for their ordered expansions. In these cases, all splitting nodes in $\mathbb{T}$ allow for the construction of a +-similarity copy of A. Fraïssé limits of classes such as $\mathcal{P}_n$, as well as the four non-trivial reducts of the rationals, also trivially have the Extension Property.

The convexly ordered equivalence relations $\mathcal{COE}_n$, as well as the more general classes $\mathcal{COE}_{n,p}$, satisfy (2) of the Extension property with $q = n + 1$. See Lemma 5.8 for a discussion. Part (2a) of the Extension property is sufficient for the proof of Theorem 5.12, and (2b) is sufficent for the proof of Theorem 6.2.

Lemma 5.7. *SFAP implies the Extension Property.*

Proof. We will actually prove a slightly stronger statement which implies (1) of the Extension Property. Let A be a subtree of some $T \in \mathcal{T}$. Without loss of generality, we may assume that either A is infinite and has infinitely many coding nodes, or else A is finite and the node in A of maximal length is a coding node. Let m either be 0, or else let m be a positive integer such that $\max(r_m(A))$ has a coding node. Let $n > m$ be least above m such that $\max(r_n(A))$ has a coding node; let c_i^A denote this coding node.

Now suppose that B is a +-similarity copy of $r_m(A)$, and suppose C is an extension of B in T such that C is +-similar to $r_{n-1}(A)$. (Such a C is easy to construct since $\mathbb{S}$ is a perfect tree whenever $\mathbf{K}$ has at least one non-trivial relation of arity greater than one.) Let X denote $\max(r_{n-1}(A))^+$, let Y denote $\max(C)^+$, and let φ be the +-similarity map from X to Y. Let t denote the node in X which extends to the coding node in $\max(r_n(A))$, and let y denote $\varphi(t)$. Extend y to some coding node $c_{i'}^T$ in T such that the substructure of $\mathbf{K}$ represented by the coding nodes in B along with $c_{i'}^T$ is isomorphic to the substructure of $\mathbf{K}$ represented by the coding nodes in $r_n(A)$.

Fix any $u \in X$ such that $u \neq t$, and let z denote $\varphi(u)$. Let c_j^T denote the least coding node in A extending u. By SFAP, there is an extension of z to some coding node $c_{j'}^T$ representing a vertex w' in $\mathbf{K}$ such that the substructure of $\mathbf{K}$ represented by the coding nodes in B along with $c_{i'}^T$ and $c_{j'}^T$ is isomorphic to the substructure of $\mathbf{K}$ represented by the coding nodes in $r_n(A)$ along with c_j^T. Let u' denote the unique extension of u in $\max(r_n(A))$, and let z' denote the truncation of $c^T_{j'}$ to the length $|c^T_{i'}|+1$. Then $(z')^+(c^T_{j'};B) \sim (u')^+(c^T_i;r_m(A))$. Therefore, the union of C along with $\{u' : u \in Y \setminus \{y\}\} \cup \{c^T_{i'}\}$ is +-similar to $r_n(A)$. It follows that the Extension Property holds. □

Lemma 5.8. *The Fraïssé limits of the Fraïssé classes $\mathcal{COE}_{n,p}$ satisfy the Extension Property.*

Proof. Fix n, p, with at least one of n, p greater than one, and denote the Fraïssé limit of $\mathcal{COE}_{n,p}$ as $\mathbf{K}$. Recall that the language of $\mathcal{COE}_{n,p}$ has one binary relation symbol $<$, interpreted in $\mathbf{K}$ as a linear order; p-many unary relation symbols, $P_0, \dots, P_{p-1}$, whose interpretations partition $\mathbf{K}$ into p-many dense pieces; and n-many binary relation symbols $E_0, \dots, E_{n-1}$, each interpreted in $\mathbf{K}$ as an equivalence relation with convexly ordered equivalence classes such that if $i < j < n$, then $E^{\mathbf{K}}_j$ coarsens $E^{\mathbf{K}}_i$. Let $v^{\mathbb{T}}_m$ denote the vertex in $\mathbf{K}$ represented by the m-th coding node, $c^{\mathbb{T}}_m$, in $\mathbb{T}$. Given a splitting node s in $\mathbb{T}$, define $\psi(s) = n$ if $\neg E_{n-1}(x, v^{\mathbb{T}}_m)$ is in s, for all $m < |s|-1$. Otherwise, define $\psi(s)$ to be the least $i < n$ such that $E_i(x, v^{\mathbb{T}}_m)$ is in s, for some $m < |s|-1$. In this second case, any coding node in $\mathbb{T}$ extending s represents a vertex which is in the same $E_{\psi(s)}$-equivalence class as some vertex $v^{\mathbb{T}}_m$ for some $m < |s|-1$. Note that if $s \subseteq t$ are splitting nodes in $\mathbb{T}$, then $\psi(s) \geq \psi(t)$. It is routine to check that $\mathcal{COE}_{n,p}$ satisfies (2) of the Extension Property with this ψ. □

By an *antichain* of coding nodes, we mean a set of coding nodes which is pairwise incomparable with respect to the tree partial order of inclusion.

Set-up for Theorem 5.12. Let T be a diagonal coding tree in $\mathcal{T}$. Fix a finite antichain of coding nodes $\tilde{C} \subseteq T$. We abuse notation and also write $\tilde{C}$ to denote the tree that its meet-closure induces in T. Let $\tilde{A}$ be a fixed proper initial segment of $\tilde{C}$, allowing for $\tilde{A}$ to be the empty set. Thus, $\tilde{A} = \tilde{C}{\downarrow}\ell$, where ℓ is the length of some splitting or coding node in $\tilde{C}$ (let $\ell = 0$ if $\tilde{A}$ is empty). Let $\ell_{\tilde{A}}$ denote this ℓ, and note that any non-empty $\max(\tilde{A})$ either has a coding node or a splitting node. Let $\tilde{x}$ denote the shortest splitting or coding node in $\tilde{C}$ with length greater than $\ell_{\tilde{A}}$, and define $\tilde{X} = \tilde{C} \restriction |\tilde{x}|$. Then $\tilde{A} \cup \tilde{X}$ is an initial segment of $\tilde{C}$; let $\ell_{\tilde{X}}$ denote $|\tilde{x}|$. There are two cases:

Case (a). $\tilde{X}$ has a splitting node.

Case (b). $\tilde{X}$ has a coding node.

Let $d+1$ be the number of nodes in $\tilde{X}$ and index these nodes as $\tilde{x}_i$, $i \leq d$, where $\tilde{x}_d$ denotes the critical node (recall that *critical node* refers to a splitting or coding node). Let

$$\tilde{B} = \tilde{C} \restriction (\ell_{\tilde{A}} + 1). \tag{23}$$

Then $\tilde{X}$ is a level set equal to or end-extending the level set $\tilde{B}$. For each $i \leq d$, define

$$\tilde{b}_i = \tilde{x}_i \restriction \ell_{\tilde{B}}. \tag{24}$$

Note that we consider nodes in $\tilde{B}$ as simply nodes to be extended; it does not matter whether the nodes in $\tilde{B}$ are coding, splitting, or neither in T.

Definition 5.9 (Weak similarity). Given finite subtrees $S, T \in \mathcal{T}$ in which each coding node is terminal, we say that S is *weakly similar* to T, and write $S \overset{w}{\sim} T$, if and only if $S \setminus \max(S) \overset{+}{\sim} T \setminus \max(T)$.

Definition 5.10 ($\mathrm{Ext}_T(B; \tilde{X})$). Let $T \in \mathcal{T}$ be fixed and let $D = r_n(T)$ for some $n < \omega$. Suppose A is a subtree of D such that $A \overset{+}{\sim} \tilde{A}$ and A is extendible to a similarity copy of $\tilde{C}$ in T. Let B be a subset of the level set $\max(D)^+$ such that B end-extends (or equals) $\max(A)^+$ and $A \cup B \overset{w}{\sim} \tilde{A} \cup \tilde{B}$. Define $\mathrm{Ext}_T(B; \tilde{X})$ to be the collection of all level sets $X \subseteq T$ such that

(1) X end-extends B;
(2) $A \cup X \overset{+}{\sim} \tilde{A} \cup \tilde{X}$; and
(3) $A \cup X$ extends to a copy of $\tilde{C}$.

For Case (b), condition (3) follows from (2). For Case (a), the Extension Property guarantees that for any level set Y end-extending B, there is a level set X end-extending Y such that $A \cup X$ satisfies condition (3).

The following theorem of Erdős and Rado will provide the pigeonhole principle for the forcing proof.

Theorem 5.11 (Erdős-Rado, [18]). *For $r < \omega$ and μ an infinite cardinal,*

$$\beth_r(\mu)^+ \to (\mu^+)^{r+1}_\mu.$$

We are now ready to prove the Ramsey theorem for level set extensions of a given finite tree.

Theorem 5.12. *Suppose that $\mathcal{K}$ has Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$, and $T \in \mathcal{T}$ is given. Let $\tilde{C}$ be a finite antichain of coding nodes in T, $\tilde{A}$ be an initial segment of $\tilde{C}$, and $\tilde{B}$ be defined as above. Suppose $D = r_n(T)$ for some $n < \omega$, and $A \subseteq D$ and $B \subseteq \max(D^+)$ satisfy $A \cup B \overset{w}{\sim} \tilde{A} \cup \tilde{B}$. Then given any coloring $h : \mathrm{Ext}_T(B; \tilde{X}) \to 2$, there is a coding tree $S \in [D, T]$ such that h is monochromatic on $\mathrm{Ext}_S(B; \tilde{X})$.*

Proof. Enumerate the nodes in B as $s_0, \dots, s_d$ so that for any $X \in \mathrm{Ext}_T(B; \tilde{X})$, the critical node in X extends s_d. Let M denote the collection of all $m \geq n$ for which there is a member of $\mathrm{Ext}_T(B; \tilde{X})$ with nodes in $T(m)$. Note that this set M is the same for any $S \in \mathcal{T}$, since $S \sim T$ for all $S, T \in \mathcal{T}$. Let $L = \{|t| : \exists m \in M\, (t \in T(m))\}$, the collection of lengths of nodes in the levels $T(m)$ for $m \in M$.

For $i \leq d$, let $T_i = \{t \in T : t \supseteq s_i\}$. Let κ be large enough, so that the partition relation $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds. The following forcing notion $\mathbb{P}$ adds κ many paths through each T_i, $i < d$, and one path through T_d.

In both Cases (a) and (b), define $\mathbb{P}$ to be the set of finite partial functions p such that

$$p : (d \times \vec{\delta}_p) \cup \{d\} \to T(m_p),$$

where

(1) $m_p \in M$ and $\vec{\delta}_p$ is a finite subset of κ;
(2) $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i(m_p)$ for each $i < d$;
(3) $p(d)$ is the critical node in $T_d(m_p)$; and
(4) For any choices of $\delta_i \in \vec{\delta}_p$, the level set $\{p(i, \delta_i) : i < d\} \cup \{p(d)\}$ is a member of $\mathrm{Ext}_T(B; \tilde{X})$.

Given $p \in \mathbb{P}$, the *range of* p is defined as

$$\mathrm{ran}(p) = \{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}.$$

Let ℓ_p denote the length of the nodes in $\mathrm{ran}(p)$. If also $q \in \mathbb{P}$ and $\vec{\delta}_p \subseteq \vec{\delta}_q$, then we let $\mathrm{ran}(q \restriction \vec{\delta}_p)$ denote $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$.

In Case (a), the partial ordering on $\mathbb{P}$ is simply reverse inclusion: $q \le p$ if and only if

(1) $m_q \ge m_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, $q(d) \supseteq p(d)$; and
(2) $q(i,\delta) \supseteq p(i,\delta)$ for each $(i,\delta) \in d \times \vec{\delta}_p$.

In Case (b), we define $q \le p$ if and only if (1) and (2) hold and additionally, the following third requirement holds:

(3) Letting $U = T \restriction (\ell_p - 1)$, $U \cup \mathrm{ran}(p) \overset{+}{\sim} U \cup \mathrm{ran}(q \restriction \delta_p)$.

(Requirement (3) is stronger than that which was used for the Rado graph in [10], because for relations of arity three or more, the extension q must preserve information about 1-types over the fixed finite structure which we wish to extend.) Then $(\mathbb{P}, \le)$ is a separative, atomless partial order.

The next part of the proof (up to and including Lemma 5.15) follows that of [10] almost verbatim. We include it for the reader's convenience. For $(i,\alpha) \in d \times \kappa$, let

$$\dot{b}_{i,\alpha} = \{\langle p(i,\alpha), p\rangle : p \in \mathbb{P} \text{ and } \alpha \in \vec{\delta}_p\}, \tag{25}$$

a $\mathbb{P}$-name for the α-th generic branch through T_i. Let

$$\dot{b}_d = \{\langle p(d), p\rangle : p \in \mathbb{P}\}, \tag{26}$$

a $\mathbb{P}$-name for the generic branch through T_d. Given a generic filter $G \subseteq \mathbb{P}$, notice that $\dot{b}_d^G = \{p(d) : p \in G\}$, which is a cofinal path of critical nodes in T_d. Let $\dot{L}_d$ be a $\mathbb{P}$-name for the set of lengths of critical nodes in $\dot{b}_d$, and note that $\mathbb{P}$ forces that $\dot{L}_d \subseteq L$. Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on $\dot{L}_d$. Given $p \in \mathbb{P}$, recall that ℓ_p denotes the lengths of the nodes in $\mathrm{ran}(p)$, and notice that

$$p \Vdash \forall (i,\alpha) \in d \times \vec{\delta}_p\, (\dot{b}_{i,\alpha} \restriction \ell_p = p(i,\alpha)) \wedge (\dot{b}_d \restriction \ell_p = p(d)). \tag{27}$$

We will write sets $\{\alpha_i : i < d\}$ in $[\kappa]^d$ as vectors $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1}\rangle$ in strictly increasing order. For $\vec{\alpha} \in [\kappa]^d$, let

$$\dot{b}_{\vec{\alpha}} = \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}}, \dot{b}_d\rangle. \tag{28}$$

For $\ell < \omega$, let

$$\dot{b}_{\vec{\alpha}} \restriction \ell = \langle \dot{b}_{0,\alpha_0} \restriction \ell, \dots, \dot{b}_{d-1,\alpha_{d-1}} \restriction \ell, \dot{b}_d \restriction \ell\rangle. \tag{29}$$

One sees that h is a coloring on level sets of the form $\dot{b}_{\vec{\alpha}} \restriction \ell$ whenever this is forced to be a member of $\mathrm{Ext}_T(B; \tilde{X})$. Given $\vec{\alpha} \in [\kappa]^d$ and $p \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_p$, let

$$X(p, \vec{\alpha}) = \{p(i,\alpha_i) : i < d\} \cup \{p(d)\}, \tag{30}$$

recalling that this level set $X(p,\vec{\alpha})$ is a member of $\mathrm{Ext}_T(B; \tilde{X})$.

For each $\vec{\alpha} \in [\kappa]^d$, choose a condition $p_{\vec{\alpha}} \in \mathbb{P}$ satisfying the following:

(1) $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$.
(2) There is an $\varepsilon_{\vec{\alpha}} \in 2$ such that $p_{\vec{\alpha}} \Vdash$ "$h(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many ℓ in $\dot{L}_d$".
(3) $h(X(p_{\vec{\alpha}}, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}$.

Such conditions can be found as follows: Fix some $X \in \mathrm{Ext}_T(B;\tilde{X})$ and let t_i denote the node in X extending s_i, for each $i \le d$. For $\vec{\alpha} \in [\kappa]^d$, define

$$p^0_{\vec{\alpha}} = \{\langle (i,\delta), t_i\rangle : i < d,\ \delta \in \vec{\alpha}\} \cup \{\langle d, t_d\rangle\}.$$

Then (1) will hold for all $p \le p^0_{\vec{\alpha}}$, since $\vec{\delta}_{p^0_{\vec{\alpha}}} = \vec{\alpha}$. Next, let $p^1_{\vec{\alpha}}$ be a condition below $p^0_{\vec{\alpha}}$ which forces $h(\dot{b}_{\vec{\alpha}} \restriction \ell)$ to be the same value for $\dot{\mathcal{U}}$ many $\ell \in \dot{L}_d$. Extend this to some condition $p^2_{\vec{\alpha}} \le p^1_{\vec{\alpha}}$ which decides a value $\varepsilon_{\vec{\alpha}} \in 2$ so that $p^2_{\vec{\alpha}}$ forces $h(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many ℓ in $\dot{L}_d$. Then (2) holds for all $p \le p^2_{\vec{\alpha}}$. If $p^2_{\vec{\alpha}}$ satisfies (3), then let $p_{\vec{\alpha}} = p^2_{\vec{\alpha}}$. Otherwise, take some $p^3_{\vec{\alpha}} \le p^2_{\vec{\alpha}}$ which forces $\dot{b}_{\vec{\alpha}} \restriction \ell \in \mathrm{Ext}_T(B;\tilde{X})$ and $h'(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon_{\vec{\alpha}}$ for some $\ell \in \dot{L}$ with $\ell_{p^2_{\vec{\alpha}}} < \ell \le \ell_{p^3_{\vec{\alpha}}}$. Since $p^3_{\vec{\alpha}}$ forces that $\dot{b}_{\vec{\alpha}} \restriction \ell$ equals $\{p^3_{\vec{\alpha}}(i,\alpha_i) \restriction \ell : i < d\} \cup \{p^3_{\vec{\alpha}}(d) \restriction \ell\}$, which is exactly $X(p^3_{\vec{\alpha}} \restriction \ell, \vec{\alpha})$, and this level set is in the ground model, it follows that $h(X(p^3_{\vec{\alpha}} \restriction \ell, \vec{\alpha})) = \varepsilon_{\vec{\alpha}}$. Let $p_{\vec{\alpha}}$ be $p^3_{\vec{\alpha}} \restriction \ell$. Then $p_{\vec{\alpha}}$ satisfies (1)–(3).

Let $\mathcal{I}$ denote the collection of all functions $\iota : 2d \to 2d$ such that for each $i < d$, $\{\iota(2i), \iota(2i+1)\} \subseteq \{2i, 2i+1\}$. For $\vec{\theta} = \langle \theta_0, \dots, \theta_{2d-1}\rangle \in [\kappa]^{2d}$, $\iota(\vec{\theta})$ determines the pair of sequences of ordinals $\langle \iota_e(\vec{\theta}), \iota_o(\vec{\theta})\rangle$, where

$$\begin{aligned} &\iota_e(\vec{\theta}) = \langle \theta_{\iota(0)}, \theta_{\iota(2)}, \dots, \theta_{\iota(2d-2))}\rangle \\ &\iota_o(\vec{\theta}) = \langle \theta_{\iota(1)}, \theta_{\iota(3)}, \dots, \theta_{\iota(2d-1)}\rangle. \end{aligned} \tag{31}$$

We now proceed to define a coloring f on $[\kappa]^{2d}$ into countably many colors. Let $\vec{\delta}_{\vec{\alpha}}$ denote $\vec{\delta}_{p_{\vec{\alpha}}}$, $k_{\vec{\alpha}}$ denote $|\vec{\delta}_{\vec{\alpha}}|$, $\ell_{\vec{\alpha}}$ denote $\ell_{p_{\vec{\alpha}}}$, and let $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ denote the enumeration of $\vec{\delta}_{\vec{\alpha}}$ in increasing order. Given $\vec{\theta} \in [\kappa]^{2d}$ and $\iota \in \mathcal{I}$, to reduce subscripts let $\vec{\alpha}$ denote $\iota_e(\vec{\theta})$ and $\vec{\beta}$ denote $\iota_o(\vec{\theta})$, and define

$$\begin{aligned} f(\iota, \vec{\theta}) = \langle &\iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, p_{\vec{\alpha}}(d), \langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}},\ \text{and } \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle. \end{aligned} \tag{32}$$

Fix some ordering of $\mathcal{I}$ and define

$$f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle. \tag{33}$$

By the Erdős-Rado Theorem 5.11, there is a subset $K \subseteq \kappa$ of cardinality $\aleph_1$ which is homogeneous for f. Take $K' \subseteq K$ so that between each two members of K' there is a member of K. Given sets of ordinals I and J, we write $I < J$ to mean that every member of I is less than every member of J. Take $K_i \subseteq K'$ be countably infinite subsets satisfying $K_0 < \cdots < K_{d-1}$.

Fix some $\vec{\gamma} \in \prod_{i<d} K_i$, and define

$$\begin{aligned} &\varepsilon^* = \varepsilon_{\vec{\gamma}},\quad k^* = k_{\vec{\gamma}},\quad t_d = p_{\vec{\gamma}}(d), \\ &t_{i,j} = p_{\vec{\gamma}}(i, \delta_{\vec{\gamma}}(j)) \text{ for } i < d,\ j < k^*. \end{aligned} \tag{34}$$

We show that the values in equation (34) are the same for any choice of $\vec{\gamma}$.

Lemma 5.13. *For all $\vec{\alpha} \in \prod_{i<d} K_i$, $\varepsilon_{\vec{\alpha}} = \varepsilon^*$, $k_{\vec{\alpha}} = k^*$, $p_{\vec{\alpha}}(d) = t_d$, and $\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle = \langle t_{i,j} : j < k^*\rangle$ for each $i < d$.*

Proof. Let $\vec{\alpha}$ be any member of $\prod_{i<d} K_i$, and let $\vec{\gamma}$ be the set of ordinals fixed above. Take $\iota \in \mathcal{I}$ to be the identity function on $2d$. Then there are $\vec{\theta}, \vec{\theta}' \in [K]^{2d}$ such that $\vec{\alpha} = \iota_e(\vec{\theta})$ and $\vec{\gamma} = \iota_e(\vec{\theta}')$. Since $f(\iota, \vec{\theta}) = f(\iota, \vec{\theta}')$, it follows that $\varepsilon_{\vec{\alpha}} = \varepsilon_{\vec{\gamma}}$,

$k_{\vec{\alpha}} = k_{\vec{\gamma}}$, $p_{\vec{\alpha}}(d) = p_{\vec{\gamma}}(d)$, and $\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle = \langle\langle p_{\vec{\gamma}}(i, \delta_{\vec{\gamma}}(j)) : j < k_{\vec{\gamma}}\rangle : i < d\rangle$. □

Let l^* denote the length of the node t_d, and notice that the node $t_{i,j}$ also has length l^*, for each $(i, j) \in d \times k^*$.

Lemma 5.14. *Given any $\vec{\alpha}, \vec{\beta} \in \prod_{i<d} K_i$, if $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then $j = k$.*

Proof. Let $\vec{\alpha}, \vec{\beta}$ be members of $\prod_{i<d} K_i$ and suppose that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$ for some $j, k < k^*$. For $i < d$, let ρ_i be the relation from among $\{<, =, >\}$ such that $\alpha_i \, \rho_i \, \beta_i$. Let ι be the member of $\mathcal{I}$ such that for each $\vec{\theta} \in [K]^{2d}$ and each $i < d$, $\theta_{\iota(2i)} \ \rho_i \ \theta_{\iota(2i+1)}$. Fix some $\vec{\theta} \in [K']^{2d}$ such that $\iota_e(\vec{\theta}) = \vec{\alpha}$ and $\iota_o(\vec{\theta}) = \vec{\beta}$. Since between any two members of K' there is a member of K, there is a $\vec{\zeta} \in [K]^d$ such that for each $i < d$, $\alpha_i \, \rho_i \, \zeta_i$ and $\zeta_i \, \rho_i \, \beta_i$. Let $\vec{\mu}, \vec{\nu}$ be members of $[K]^{2d}$ such that $\iota_e(\vec{\mu}) = \vec{\alpha}$, $\iota_o(\vec{\mu}) = \vec{\zeta}$, $\iota_e(\vec{\nu}) = \vec{\zeta}$, and $\iota_o(\vec{\nu}) = \vec{\beta}$. Since $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, the pair $\langle j, k\rangle$ is in the last sequence in $f(\iota, \vec{\theta})$. Since $f(\iota, \vec{\mu}) = f(\iota, \vec{\nu}) = f(\iota, \vec{\theta})$, also $\langle j, k\rangle$ is in the last sequence in $f(\iota, \vec{\mu})$ and $f(\iota, \vec{\nu})$. It follows that $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\zeta}}(k)$ and $\delta_{\vec{\zeta}}(j) = \delta_{\vec{\beta}}(k)$. Hence, $\delta_{\vec{\zeta}}(j) = \delta_{\vec{\zeta}}(k)$, and therefore j must equal k. □

For each $\vec{\alpha} \in \prod_{i<d} K_i$, given any $\iota \in \mathcal{I}$, there is a $\vec{\theta} \in [K]^{2d}$ such that $\vec{\alpha} = \iota_o(\vec{\alpha})$. By the second line of equation (32), there is a strictly increasing sequence $\langle j_i : i < d\rangle$ of members of k^* such that $\delta_{\vec{\gamma}}(j_i) = \alpha_i$. By homogeneity of f, this sequence $\langle j_i : i < d\rangle$ is the same for all members of $\prod_{i<d} K_i$. Then letting t_i^* denote t_{i,j_i}, one sees that

$$p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j_i)) = t_{i,j_i} = t_i^*. \tag{35}$$

Let t_d^* denote t_d.

Lemma 5.15. *For any finite subset $\vec{J} \subseteq \prod_{i<d} K_i$, $p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a member of $\mathbb{P}$ which is below each $p_{\vec{\alpha}}$, $\vec{\alpha} \in \vec{J}$.*

Proof. Given $\vec{\alpha}, \vec{\beta} \in \vec{J}$, if $j, k < k^*$ and $\delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)$, then j and k must be equal, by Lemma 5.14. Then Lemma 5.13 implies that for each $i < d$,

$$p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) = t_{i,j} = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(j)) = p_{\vec{\beta}}(i, \delta_{\vec{\beta}}(k)). \tag{36}$$

Hence, for all $\delta \in \vec{\delta}_{\vec{\alpha}} \cap \vec{\delta}_{\vec{\beta}}$ and $i < d$, $p_{\vec{\alpha}}(i, \delta) = p_{\vec{\beta}}(i, \delta)$. Thus, $p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a function with domain $\vec{\delta}_{\vec{J}} \cup \{d\}$, where $\vec{\delta}_{\vec{J}} = \bigcup\{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$; hence , $p_{\vec{J}}$ is a member of $\mathbb{P}$. Since for each $\vec{\alpha} \in \vec{J}$, $\mathrm{ran}(p_{\vec{J}} \restriction \vec{\delta}_{\vec{\alpha}}) = \mathrm{ran}(p_{\vec{\alpha}})$, it follows that $p_{\vec{J}} \le p_{\vec{\alpha}}$ for each $\vec{\alpha} \in \vec{J}$. □

This ends the material drawn directly from [10].

We now proceed to build a (diagonal coding) tree $S \in [D, T]$ so that the coloring h will be monochromatic on $\mathrm{Ext}_S(B; \tilde{X})$. Recall that n is the integer such that $D = r_n(T)$. Let $\{m_j : j < \omega\}$ be the strictly increasing enumeration of M, noting that $m_0 \geq n$. For each $i \leq d$, extend the node $s_i \in B$ to the node t_i^*. Extend each node u in $\max(D)^+ \setminus B$ to some node u^* in $T \restriction \ell^*$. If $\tilde{X}$ has a coding node and $m_0 = n$, require also that $(u^*)^+(u^*; D) \sim u^+(u; D)$; SDAP ensures that such u^* exist. Set

$$U^* = \{t_i^* : i \leq d\} \cup \{u^* : u \in \max(D)^+ \setminus B\} \tag{37}$$

and note that U^* end-extends $\max(D)^+$.

If $m_0 = n$, then $D \cup U^*$ is a member of $r_{m_0+1}[D, T]$. In this case, let $U_{m_0+1} = D \cup U^*$, and let U_{m_1} be any member of $r_{m_1}[U_{m_0+1}, T]$. Note that U^* is the only member of $\mathrm{Ext}_{U_{m_1}}(B; \tilde{X})$, and it has h-color ε^*. Otherwise, $m_0 > n$. In this case, take some $U_{m_0} \in r_{m_0}[D, T]$ such that $\max(U_{m_0})$ end-extends U^*, and notice that $\mathrm{Ext}_{U_{m_0}}(B; \tilde{X})$ is empty.

Now assume that $j < \omega$ and we have constructed $U_{m_j} \in r_{m_j}[D, T]$ so that every member of $\mathrm{Ext}_{U_{m_j}}(B; \tilde{X})$ has h-color ε^*. Fix some $V \in r_{m_j+1}[U_{m_j}, T]$ and let $Y = \max(V)$. We will extend the nodes in Y to construct $U_{m_j+1} \in r_{m_j+1}[U_{m_j}, T]$ with the property that all members of $\mathrm{Ext}_{U_{m_j+1}}(B; \tilde{X})$ have the same h-value ε^*. This will be achieved by constructing the condition $q \in \mathbb{P}$, below, and then extending it to some condition $r \le q$ which decides that all members of $\mathrm{Ext}_T(B; \tilde{X})$ coming from the nodes in $\mathrm{ran}(r)$ have h-color ε^*.

Let $q(d)$ denote the splitting node or coding node in Y and let $\ell_q = |q(d)|$. For each $i < d$, let Y_i denote $Y \cap T_i$. For each $i < d$, take a set $J_i \subseteq K_i$ of size $\mathrm{card}(Y_i)$ and label the members of Y_i as $\{z_\alpha : \alpha \in J_i\}$. Let $\vec{J}$ denote $\prod_{i<d} J_i$. By Lemma 5.15, the set $\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is compatible, and $p_{\vec{J}} := \bigcup\{p_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$ is a condition in $\mathbb{P}$.

Let $\vec{\delta}_q = \bigcup\{\vec{\delta}_{\vec{\alpha}} : \vec{\alpha} \in \vec{J}\}$. For $i < d$ and $\alpha \in J_i$, define $q(i, \alpha) = z_\alpha$. It follows that for each $\vec{\alpha} \in \vec{J}$ and $i < d$,

$$q(i, \alpha_i) \supseteq t_i^* = p_{\vec{\alpha}}(i, \alpha_i) = p_{\vec{J}}(i, \alpha_i), \tag{38}$$

and

$$q(d) \supseteq t_d^* = p_{\vec{\alpha}}(d) = p_{\vec{J}}(d). \tag{39}$$

For $i < d$ and $\delta \in \vec{\delta}_q \setminus J_i$, we need to extend each node $p_{\vec{J}}(i, \delta)$ to some node of length ℓ_q in order to construct a condition q extending $p_{\vec{J}}$. These nodes will not be a part of the construction of U_{m_j+1}, however; they only are only a technicality allowing us to find some $r \le q \le p_{\vec{J}}$ from which we will build U_{m_j+1}. In Case (a), let $q(i, \delta)$ be any extension of $p_{\vec{J}}(i, \delta)$ in T of length ℓ_q. In Case (b), let $q(i, \delta)$ be any extension of $p_{\vec{J}}(i, \delta)$ in T of length ℓ_q with

$$q(i, \delta)^+(q(d); T \restriction (\ell^* - 1)) \sim p_{\vec{J}}(i, \delta)^+(p_{\vec{J}}(d); T \restriction (\ell^* - 1)). \tag{40}$$

The SDAP guarantees the existence of such $q(i, \delta)$. Define

$$q = \{q(d)\} \cup \{\langle (i, \delta), q(i, \delta) \rangle : i < d,\ \delta \in \vec{\delta}_q\}. \tag{41}$$

This q is a condition in $\mathbb{P}$, and $q \le p_{\vec{J}}$.

Now take an $r \le q$ in $\mathbb{P}$ which decides some ℓ_j in $\dot{L}_d$ for which $h(\dot{b}_{\vec{\alpha}} \restriction \ell_j) = \varepsilon^*$, for all $\vec{\alpha} \in \vec{J}$. This is possible since for all $\vec{\alpha} \in \vec{J}$, $p_{\vec{\alpha}}$ forces $h(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon^*$ for $\dot{\mathcal{U}}$ many $\ell \in \dot{L}_d$. By the same argument as in creating the conditions $p_{\vec{\alpha}}$, we may assume that the nodes in the image of r have length ℓ_j. Since r forces $\dot{b}_{\vec{\alpha}} \restriction \ell_j = X(r, \vec{\alpha})$ for each $\vec{\alpha} \in \vec{J}$, and since the coloring h is defined in the ground model, it follows that $h(X(r, \vec{\alpha})) = \varepsilon^*$ for each $\vec{\alpha} \in \vec{J}$. Let

$$Y_0 = \{q(d)\} \cup \{q(i, \alpha) : i < d,\ \alpha \in J_i\}, \tag{42}$$

and let

$$Z_0 = \{r(d)\} \cup \{r(i, \alpha) : i < d,\ \alpha \in J_i\}. \tag{43}$$

Now we consider the two cases separately. In Case (a), let Z be the level set consisting of the nodes in Z_0 along with a node z_y in $T\restriction\ell_j$ extending y, for each $y\in Y\setminus Y_0$. Then Z end-extends Y. By SDAP, it does not matter how the nodes z_y are chosen. Letting $U_{m_j+1}=U_{m_j}\cup Z$, we see that U_{m_j+1} is a member of $r_{m_j+1}[U_{m_j},T]$ such that h has value ε^* on $\mathrm{Ext}_{U_{m_j+1}}(B;\tilde{X})$.

In Case (b), $r(d)$ is a coding node. Since $r\le q$, the nodes in $\mathrm{ran}(r\restriction\delta_q)$ have the same passing types over $T{\downarrow}\ell_q$ as the nodes in $\mathrm{ran}(q)$ have over $T{\downarrow}\ell_q$. We now need to extend all the other members of $Y\setminus Y_0$ to nodes with the required passing types at $r(d)$. For each $y\in Y\setminus Y_0$, choose a member $z_y\supset y$ in $T_d\restriction\ell_j$ so that

$$z_y^+(r(d);U_{m_j})\sim y^+(q(d);U_{m_j}). \tag{44}$$

SDAP ensures the existence of such z_y. Let Z be the level set consisting of the nodes in Z_0 along with the nodes z_y for $y\in Y\setminus Y_0$. Then Z end-extends Y and moreover, $U_{m_j}\cup Z\overset{+}{\sim}V$. Letting $U_{m_j+1}=U_{m_j}\cup Y$, we see that U_{m_j+1} is a member of $r_{m_j+1}[U_{m_j},T]$ and h has value ε^* on $\mathrm{Ext}_{U_{m_j+1}}(B;\tilde{X})$.

Now that we have constructed U_{m_j+1}, let $U_{m_{j+1}}$ be any member of $r_{m_{j+1}}[U_{m_j+1},T]$. This completes the inductive construction. Let $S=\bigcup_{j<\omega}U_{m_j}$. Then S is a member of $[D,T]$ and for each $X\in\mathrm{Ext}_S(B)$, $h(X)=\varepsilon^*$. Thus, S satisfies the theorem. $\square$

Remark 5.16. By the construction in the previous proof, in Case (b) the coding nodes in any member $X\in\mathrm{Ext}_S(B;\tilde{X})$ extend the coding node t_d^*. It then follows from (3) in Definition 4.21 that for every level set $X\subseteq S$ with $A\cup X\sim\tilde{A}\cup\tilde{X}$, the coding node c in X automatically satisfies $c^+(c;A)\sim(t_d^*)^+(t_d^*;A)\sim\tilde{x}_d^+(\tilde{x}_d;\tilde{A})$, where $\tilde{x}_d$ denotes the coding node in $\tilde{X}$. Thus, $A\cup X\overset{+}{\sim}\tilde{A}\cup\tilde{X}$ if and only if the non-coding nodes in X have immediate successors with similar passing types over $A\cup\{c\}$ as their counterparts in $\tilde{X}$ have over $\tilde{A}\cup\{\tilde{x}_d\}$. This observation will be useful in the proof of next theorem.

Recall that two antichains of coding nodes are considered similar if the trees induced by their meet-closures are similar. The next theorem will yield a Ramsey theorem for finite diagonal antichains of coding nodes.

Theorem 5.17. *Suppose that $\mathcal{K}$ has a Fraïssé limit satisfying SDAP$^+$ and that $\mathbb{T}$ is a diagonal coding subtree of $\mathbb{U}(\mathbf{K})$. Let $\tilde{C}\subseteq\mathbb{T}$ be an antichain of coding nodes, and let $T\in\mathcal{T}$ be fixed. Given any coloring of the set $\{C\subseteq T: C\sim\tilde{C}\}$, there is an $S\le T$ such that all members of $\{C\subseteq S: C\sim\tilde{C}\}$ have the same color.*

Proof. The proof is by reverse induction on the levels in $\tilde{C}$. Suppose that $\tilde{C}$ has $n\ge 1$ levels, and let $\tilde{X}$ denote $\tilde{C}\restriction\ell_{\tilde{C}}$, the maximum level of $\tilde{C}$. Then $\tilde{X}$ is a single coding node. Let $\tilde{A}$ denote $\tilde{C}\setminus\tilde{X}$; that is, $\tilde{A}$ is the initial segment of all but the maximum level of $\tilde{C}$.

If $\tilde{C}$ is a single coding node (i.e. $n=1$), then let $A=D=\emptyset$ and let B be the singleton $T(0)$, which must be an initial segment of any similarity copy of $\tilde{X}$ in T. Then Theorem 5.12 implies that there is an $S\in[\emptyset,T]$ (i.e. $S\le T$) such that all members of $\mathrm{Ext}_S(B;\tilde{X})$ have the same color. Since all coding nodes similar to $\tilde{X}$ must end-extend B, it follows that every similarity copy of $\tilde{C}$ in S has the same color.

Now assume that $\tilde{C}$ has n levels, where $n\ge 2$. Let m_0 be the least integer such that $r_{m_0}(T)$ contains a $+$-similarity copy of $\tilde{A}$ extending to a copy of $\tilde{C}$, and let

$D_0 = r_{m_0}(T)$. Let $A^0, \dots, A^j$ list those $A \subseteq T$ such that $\max(A) \subseteq \max(D_0)$ and A extends to a similarity copy of $\tilde{C}$. For $i \le j$, let B^i denote $(A^i)^+$, which we recall is the tree consisting of the nodes in A^i along with all immediate successors of nodes in $\max(A^i)$. (These immediate successors are the same whether we consider them in $\mathbb{T}$ or in T.) Each B^j is a subtree of $(D_0)^+$. Apply Theorem 5.12 to obtain a $T_0^0 \in [D_0, T]$ such that h is monochromatic on $\mathrm{Ext}_{T_0^0}(B^0; \tilde{X})$. Repeat this process, each time thinning the previous tree to obtain $T_0^{i+1} \in [D_0, T_0^i]$ so that for each $i \le j$, $\mathrm{Ext}_{T_0^j}(B^i; \tilde{X})$ is monochromatic. Let T_0 denote T_0^j. Then for each A^i, $i \le j$, every extension of A^i to a similarity copy of $\tilde{A} \cup \tilde{X}$ inside T_0 has the same color.

Given $k < \omega$ and T_k, let m_{k+1} be the least integer greater than m_k such that $r_{m_{k+1}}(T_k)$ contains a +-similarity copy of $\tilde{A}$ extending to a copy of $\tilde{C}$. Let $D_{k+1} = r_{m_{k+1}}(T_k)$, and index those A with $\max(A) \subseteq \max(D_{k+1})$ such that A extends to a similarity copy of $\tilde{C}$ as A^i, $i \le j$ for some j. Repeat the above process applying Theorem 5.12 finitely many times to obtain a $T_{k+1} \in [D_{k+1}, T_k]$ with the property that for each $i \le j$, all similarity copies of $\tilde{A} \cup \tilde{X}$ in T_{k+1} extending A^i have the same color.

Since each T_{k+1} is a member of $[D_{k+1}, T_k]$, the union $\bigcup_{k<\omega} D_k$ is a member of $\mathcal{T}$, call it S_1. This induces a well-defined coloring of the copies of $\tilde{A}$ in S_1 as follows: Given $A \subseteq S_1$ a similarity copy of $\tilde{A}$ extending to a copy of $\tilde{C}$, let k be least such that A is contained in $r_{m_k}(S_1)$. Then $\max(A)$ is contained in $\max(D_k)$, and $S_1 \in [D_k, T_k]$ implies that for each level set extension X of A in S_1 such that $A \cup X \sim \tilde{A} \cup \tilde{X}$, these similarity copies of $\tilde{C}$ have the same color.

This now induces a coloring on +-similarity copies of $\tilde{A}$ inside S_1. Let $\tilde{C}_{n-1}$ denote this $\tilde{A}$, $\tilde{X}_{n-1}$ denote $\max(\tilde{C}_{n-1})$, and $\tilde{A}_{n-1}$ denote $\tilde{C}_{n-1} \setminus \tilde{X}_{n-1}$. Repeat the argument in the previous three paragraphs to obtain $S_2 \le S_1$ such that for each +-similarity copy of A_{n-1} in S_2, all extensions to +-similarity copies of $\tilde{C}_{n-1}$ in S_2 have the same color.

At the end of the reverse induction, we obtain an $S := S_n \le T$ such that all similarity copies of $\tilde{C}$ in S have the same color. □

The previous theorem immediately yields the following corollary for the special case when $\tilde{C}$ is a single coding node. Recall from Definition 2.3 that a Fraïssé structure $\mathbf{K}$ is said to be *indivisible* when every singleton substructure of $\mathbf{K}$ has big Ramsey degree equal to one.

Corollary 5.18. *If the Fraïssé limit* $\mathbf{K}$ *of a Fraïssé class* $\mathcal{K}$ *satisfies SDAP*$^+$*, then* $\mathbf{K}$ *is indivisible.*

Remark 5.19. Recall Convention 5.1 that if (a) $\mathcal{K}$ satisfies SFAP, or (b) $\mathbf{K}$ satisfies SDAP$^+$ and either has no unary relations or has no transitive relations, then we work inside a diagonal coding subtree $\mathbb{T}$ of $\mathbb{S}$. Otherwise, we have a transitive relation as well as unary relations and we work inside $\mathbb{U}$. In this case, any subset $D \subseteq \mathbb{U}$ for which $\mathbf{K} \restriction D$ contains a copy of $\mathbf{K}$ will contain a subset $D' \subseteq D$ such that $\mathbf{K} \restriction D' \cong \mathbf{K}$ and for each non-terminal node $s \in D'$ and each $\gamma \in \Gamma$, there is a coding node $c \in D'$ extending s such that $\gamma(v)$ holds in $\mathbf{K}$, where v is the vertex of $\mathbf{K}$ represented by c.

The next lemma shows that if $\mathcal{K}$ has Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$, then within any diagonal coding tree, there is an antichain of coding nodes representing a copy of $\mathbf{K}$.

Lemma 5.20. *Suppose Fraïssé class $\mathcal{K}$ has Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$. If $\mathcal{K}$ satisfies SFAP or $\mathbf{K}$ either has no transitive relation or has no unary relations, let T be a diagonal coding subtree of $\mathbb{S}(\mathbf{K})$; otherwise, let T be a diagonal coding subtree of $\mathbb{U}(\mathbf{K})$. Then there is an infinite antichain of coding nodes $\mathbb{D} \subseteq T$ so that $\mathbf{K}\restriction\mathbb{D} \cong^\omega \mathbf{K}$.*

Proof. We will use $c_n^{\mathbb{D}}$ to denote the n-th coding node in $\mathbb{D}$, and $v_n^{\mathbb{D}}$ to denote the vertex in $\mathbf{K}$ coded by $c_n^{\mathbb{D}}$. The antichain $\mathbb{D}$ will look almost exactly like T in the following sense: For each n, the level set of $\mathbb{D}$ containing the n-th coding node, denoted $\mathbb{D}\restriction|c_n^{\mathbb{D}}|$, will have exactly one more node than $T\restriction|c_n^T|$, and the $\prec$-preserving bijection between $T\restriction|c_n^T|$ and $(\mathbb{D}\restriction|c_n^{\mathbb{D}}|)\setminus\{c_n^{\mathbb{D}}\}$ will preserve passing types of the immediate successors. (This is not necessary to the results on big Ramsey degrees, but since we can do this, we will.) Moreover, letting T' be the coding tree obtained by deleting the coding nodes in $\mathbb{D}$ and declaring the node t in $(\mathbb{D}\restriction|c_n^{\mathbb{D}}|)\setminus\{c_n^{\mathbb{D}}\}$ which has $t\wedge c_n^{\mathbb{D}}$ of maximal length to be the n-th coding node in T', then $T'\sim T$.

Let m_n denote the integer such that the n-th coding node in T is in the m_n-th level of T; that is, c_n^T is in the maximal level of $r_{m_n+1}(T)$. To construct $\mathbb{D}$, begin by taking the first m_0 levels of $\mathbb{D}$ to equal those of T; that is, let $r_{m_0}(\mathbb{D}) = r_{m_0}(T)$. Each of these levels contains a splitting node. Let X denote the set of immediate successors in $\widehat{T}$ of the maximal nodes in $r_{m_0}(\mathbb{D})$. By SDAP, whatever we choose to be $c_0^{\mathbb{D}}$, each node in X can extend to a node in T with the desired passing type at $c_0^{\mathbb{D}}$.

Let s denote the node in X which extends to c_0^T. It only remains to find a splitting node extending s whose immediate successors can be extended to a coding node $c_0^{\mathbb{D}}$ (which will be terminal in $\mathbb{D}$) and another node z of length $|c_0^{\mathbb{D}}|+1$ satisfying

$$z(c_0^{\mathbb{D}}) \sim c_0^+(c_0); \tag{45}$$

or in other words, $z\restriction(\mathbf{K}\restriction\{v_0^{\mathbb{D}}\})$ is the same as the type of c_0^+ over $\mathbf{K}\restriction\{v_0\}$.

To do this, we utilize SDAP: In this application of SDAP, $\mathbf{A}$ is the empty structure and $\mathbf{C}$ is the structure $\mathbf{K}\restriction\{v_0^T, v_i^T\}$ for any $i>0$ such that c_i^T extends c_0^T. Extend s to some splitting node $s' \in T$ long enough so that the structure $\mathbf{K}\restriction(T\restriction|s'|)$ acts as $\mathbf{A}'$ as in the set-up of (B) in SDAP. In (B1), we take $\mathbf{C}'$ to be a copy of $\mathbf{C}$ represented by some coding nodes c_j^T, c_k^T, where $s' \subseteq c_j^T \subseteq c_k^T$. In (B2), we let $\mathbf{B} = \mathbf{A}'$, and take $\sigma = \tau = s'$. Let t_0, t_1 denote the immediate successors of s' in $\widehat{T}$, and take a coding node in T, which we denote $c_0^{\mathbb{D}}$, extending t_0. (The vertex $v_0^{\mathbb{D}}$ which $c_0^{\mathbb{D}}$ represents is the v'' in (B3).) Then by SDAP, there is a coding node c_m^T extending t_1 such that

$$c_m^T(c_0^{\mathbb{D}}) \sim c_0^+(c_0). \tag{46}$$

We let $y = c_m^T\restriction|c_0^{\mathbb{D}}|$ and $z = c_m^T\restriction(|c_0^{\mathbb{D}}|+1)$. The passing type of z at $c_0^{\mathbb{D}}$ is the desired passing type. We let $\mathbb{D}\restriction(|c_0^{\mathbb{D}}|+1)$ consist of the node z along with extensions of the nodes in $X\setminus\{s\}$ to the length of z so that their passing types at $c_0^{\mathbb{D}}$ are as desired; that is, the $\prec$-preserving bijection between $T\restriction(|c_0^T|+1)$ and $\mathbb{D}\restriction(|c_0^{\mathbb{D}}|+1)$ preserves passing types at c_0^T and $c_0^{\mathbb{D}}$, respectively. We let $\mathbb{D}\restriction|c_0^{\mathbb{D}}|$ equal $\{c_0^{\mathbb{D}}\}\cup\mathbb{D}\restriction|c_0^{\mathbb{D}}|$.

For the general construction stage, given $\mathbb{D}$ up to the level of $|c_n^{\mathbb{D}}|+1$, let X denote the level set $\mathbb{D}\restriction(|c_n^{\mathbb{D}}|+1)$. Extend the nodes in X in the same way that the nodes in $T\restriction|c_{n+1}^T|$ extend the nodes in $T\restriction(|c_n^T|+1)$. Let s denote the node in X which needs to be extended to the next coding node $c_{n+1}^{\mathbb{D}}$, and repeat the argument above find a suitable splitting node and extensions to a coding node $c_{n+1}^{\mathbb{D}}$ as well as a non-coding node of the same height with the desired passing type at $c^{\mathbb{D}}_{n+1}$ over $\{c^{\mathbb{D}}_i : i \leq n\}$. By SDAP, the other nodes in X extend to have the desired passing types. $\square$

By Remark 5.16, given two antichains of coding nodes C and C', it follows that $C \sim C'$ if and only if for any k, the first k levels of the trees induced by C and C', respectively, are $+$-similar.

Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$. If $\mathcal{K}$ satisfies SFAP, or $\mathbf{K}$ either has no transitive relation or has no unary relations, then let $\mathbb{T}$ be a diagonal coding subtree of $\mathbb{S}(\mathbf{K})$. Otherwise, let $\mathbb{T}$ be a diagonal coding subtree of $\mathbb{U}(\mathbf{K})$. Recalling Definition 4.15 and recalling that we may identify a subset of $\mathbb{T}$ with the subtree it induces, given an antichain of coding nodes $C \subseteq \mathbb{T}$, we let $\mathrm{Sim}(C)$ denote the set of all antichains C' of coding nodes in $\mathbb{T}$ such that $C' \sim C$. Thus, $\mathrm{Sim}(C)$ is a $\sim$-equivalence class, and we call $\mathrm{Sim}(C)$ a *similarity type*. For $S \subseteq \mathbb{T}$, we write $\mathrm{Sim}_S(C)$ for the set of $C' \subseteq S$ such that $C' \sim C$.

Definition 5.21. We say that C *represents* a copy of a structure $\mathbf{G} \in \mathcal{K}$ when $\mathbf{K} \restriction C \cong \mathbf{G}$. Given $\mathbf{G} \in \mathcal{K}$, let $\mathrm{Sim}(\mathbf{G})$ denote a set consisting of one representative from each similarity type $\mathrm{Sim}(C)$ of diagonal antichains of coding nodes $C \subseteq \mathbb{T}$ representing $\mathbf{G}$.

The next theorem providing upper bounds follows immediately from Theorem 5.17 and Lemma 5.20.

Theorem 5.22 (Upper Bounds)**.** *Suppose $\mathcal{K}$ is a Fraïssé class with Fraisse limit $\mathbf{K}$ satisfying SDAP$^+$. Then for each $\mathbf{G} \in \mathcal{K}$, the big Ramsey degree of $\mathbf{G}$ in $\mathbf{K}$ is bounded by the number of similarity types of diagonal antichains of coding nodes representing $\mathbf{G}$; that is,*

$$T(\mathbf{G}, \mathbf{K}) \leq |\,\mathrm{Sim}(\mathbf{G})|.$$

Moreover, given any finite collection $\mathcal{G}$ of structures in $\mathcal{K}$ and any coloring of all copies of each $\mathbf{G} \in \mathcal{G}$ in $\mathbf{K}$ into finitely many colors, there is a substructure $\mathbf{J}$ of $\mathbf{K}$ such that $\mathbf{J} \cong^{\omega} \mathbf{K}$ and each $\mathbf{G} \in \mathcal{G}$ takes at most $|\,\mathrm{Sim}(\mathbf{G})|$ many colors in $\mathbf{J}$.

Proof. Let $\mathcal{G}$ be a finite collection of structures in $\mathcal{K}$. Given any $T \in \mathcal{T}$, apply Theorem 5.17 finitely many times to obtain a coding subtree $S \leq T$ such that the coloring takes one color on the set $\mathrm{Sim}_S(C)$, for each $C \in \bigcup\{\mathrm{Sim}(\mathbf{G}) : \mathbf{G} \in \mathcal{G}\}$. Then apply Lemma 5.20 to take an antichain of coding nodes, $\mathbb{D} \subseteq S$, such that $\mathbf{K} \restriction \mathbb{D} \cong^{\omega} \mathbf{K}$. Letting $\mathbf{J} = \mathbf{K} \restriction \mathbb{D}$, we see that there are at most $|\,\mathrm{Sim}(\mathbf{G})|$ many colors on the copies of $\mathbf{G}$ in $\mathbf{J}$. $\square$

In the next section, we will show that these bounds are exact.

6. Simply characterized big Ramsey degrees and structures

In this section we prove that if a Fraïssé limit $\mathbf{K}$ of a Fraïssé class $\mathcal{K}$ satisfies SDAP$^+$, then we can characterize the exact big Ramsey degrees of $\mathbf{K}$; furthermore, $\mathbf{K}$ admits a big Ramsey structure. We first show, in Theorem 6.2, that each of the similarity types in Theorem 5.22 persists, and hence these similarity types form canonical partitions. From this, we obtain a succinct characterization of the exact big Ramsey degrees of $\mathbf{K}$. We then prove, in Theorem 6.9, that canonical partitions characterized via similarity types satisfy a condition of Zucker ([46]) guaranteeing the existence of big Ramsey structures. This involves showing how

Zucker's condition, which is phrased in terms of colorings of embeddings of a given structure, can be met by canonical partitions that are in terms of colorings of copies of a given structure. The big Ramsey structure for $\mathbf{K}$ thus obtained also has a simple characterization. From these results, we deduce the Main Theorem in Section 1.

Recall from Definition 2.4 the notion of *persistence*. We first show, in Theorem 6.2, that given $\mathbf{G} \in \mathcal{K}$, each of the similarity types in $\mathrm{Sim}(\mathbf{G})$ persists in any subcopy of $\mathbf{K}$. From this, it will follow that the big Ramsey degree $T(\mathbf{G}, \mathcal{K})$ is exactly the cardinality of $\mathrm{Sim}(\mathbf{G})$ (Theorem 6.7). The proof of Theorem 6.2 follows the outline and many ideas of the proof of Theorem 4.1 in [30], where Laflamme, Sauer, and Vuksanovic proved persistence of diagonal antichains for unconstrained binary relational structures.

Recall that Γ denotes the set of all complete 1-types of elements of $\mathbf{K}$ over the empty set. For $\gamma \in \Gamma$, we let $\mathbb{C}_\gamma$ denote the set of coding nodes c_n in $\mathbb{S}$ such that $\gamma(v_n)$ holds in $\mathbf{K}$, where v_n is the vertex of $\mathbf{K}$ represented by c_n; let γ_{c_n} denote this γ. The next definition extends the notion of "passing number preserving map" from Theorem 4.1 in [30].

Definition 6.1. Given two subsets $S, T \subseteq \mathbb{S}$ with coding nodes $\langle c^S_n : n < M \rangle$ and $\langle c^T_n : n < N \rangle$, respectively, where $M \leq N \leq \omega$, we say that a map $\varphi : S \to T$ is *passing type preserving (ptp)* if and only if the following hold:

1. $|s| < |t|$ implies that $|\varphi(s)| < |\varphi(t)|$.
2. φ takes each coding node in S to a coding node in T, and $\gamma_{\varphi(c^S_n)} = \gamma_{c^S_n}$ for each $n \leq M$.
3. φ preserves passing types: For any $s \in S$ and $m < M$ with $|c^S_{m-1}| < |s|$, $\varphi(s)(\varphi(c^S_m); \{\varphi(c^S_0), \dots \varphi(c^S_{m-1})\}) \sim s(c^S_m; \{c^S_0, \dots, c^S_{m-1}\})$.

Theorem 6.2 (Persistence)**.** *Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{K}$ an enumerated Fraïssé structure for $\mathcal{K}$. Suppose that $\mathbf{K}$ satisfies SDAP$^+$. Let $\mathbb{T}$ be a diagonal coding tree representing a copy of $\mathbf{K}$, let $\mathbb{D} \subseteq \mathbb{T}$ be any antichain of coding nodes representing $\mathbf{K}$, and let A be any antichain of coding nodes in $\mathbb{D}$. Then for any subset $D \subseteq \mathbb{D}$ representing a copy of $\mathbf{K}$, there is a similarity copy of A in D; that is, A persists in D.*

Proof. We shall be working under the assumption that either (a) there is an antichain of coding nodes $\mathbb{D} \subseteq \mathbb{T} \subseteq \mathbb{S}$ such that $\mathbf{K} \restriction \mathbb{D} \cong \mathbf{K}$, or (b) that for every antichain of coding nodes $\mathbb{D} \subseteq \mathbb{T} \subseteq \mathbb{U}$ such that $\mathbf{K} \restriction \mathbb{D} \cong \mathbf{K}$, there is a subset D also coding $\mathbf{K}$ with the property that for each non-terminal node $t \in D$ and for each $\gamma \in \Gamma$, there is a coding node in $D \cap \mathbb{C}_\gamma$ extending t. (Recall Remark 5.19.) In either case, we let $\mathbb{D}$ be an antichain of coding nodes in $\mathbb{T}$ representing a copy of $\mathbf{K}$, where $\mathbb{D}$ is constructed as in Lemma 5.20. Throughout, we shall use the notation $\mathbb{U}$, but keep in mind that if (a) above holds, then we are working in $\mathbb{S}$.

Without loss of generality, we may assume that $\mathbf{K} \restriction \mathbb{D} \cong^\omega \mathbf{K}$, by thinning $\mathbb{D}$ if necessary. Let $D \subseteq \mathbb{D}$ be any subset such that $\mathbf{K} \restriction D \cong \mathbf{K}$; let $\mathbf{J}$ denote $\mathbf{K} \restriction D$. Again, without loss of generality, we may assume that $\mathbf{J} \cong^\omega \mathbf{K}$. Let $\mathbb{C} = \{c_n : n < \omega\}$ denote the set of all coding nodes in $\mathbb{U}$, and note that $D \subseteq \mathbb{D} \subseteq \mathbb{C}$. Then the map $\varphi : \mathbb{C} \to D$ via $\varphi(c_n) = c^D_n$ is passing type preserving, where $\langle c^D_n : n < \omega \rangle$ is the enumeration of the nodes in D in order of increasing length.

Define

$$\overline{D} = \{c^D_n \restriction |c^D_m| : m \leq n < \omega\}. \tag{47}$$

Then $\overline{D}$ is a union of level sets (but is not meet-closed). We extend the map φ to a map $\bar{\varphi}: \mathbb{U} \to \overline{D}$ as follows: Given $s \in \mathbb{U}$, let n be least such that $c_n \supseteq s$ and m be the integer such that $|s| = |c_m|$, and define $\bar{\varphi}(s) = \varphi(c_n) \restriction |\varphi(c_m)|$; in other words, $\bar{\varphi}(s) = c_n^D \restriction |c_m^D|$.

Lemma 6.3. *$\bar{\varphi}$ is passing type preserving.*

Proof. For $s \in \mathbb{U}(m)$, let $n > m$ be least such that $s = c_n \restriction |c_m|$. Then for any for $i < m$,

$$\begin{aligned}\bar{\varphi}(s)(\varphi(c_i);\{\varphi(c_0),\dots,\varphi(c_{i-1})\}) &= (\varphi(c_n)\restriction|\varphi(c_m)|)(\varphi(c_i);\{\varphi(c_0),\dots,\varphi(c_{i-1})\})\\ &= (c_n^D\restriction|c_m^D|)(c_i^D;\{c_0^D,\dots,c_{i-1}^D\})\\ &= c_n^D(c_i^D;\{c_0^D,\dots,c_{i-1}^D\})\\ &\sim c_n(c_i;\{c_0,\dots,c_{i-1}\})\\ &= (c_n\restriction|c_m|)(c_i;\{c_0,\dots,c_{i-1}\})\\ &= s(c_i;\{c_0,\dots,c_{i-1}\}) \qquad (48)\end{aligned}$$

where the $\sim$ holds since $\varphi : \mathbb{C} \to D$ is ptp. Therefore, $\bar{\varphi}$ is ptp. □

Given a fixed subset $S \subseteq \mathbb{U}$ and $s \in S$, we let $\widehat{s}$ denote the set of all $t \in S$ such that $t \supseteq s$. The ambient set S will either be $\mathbb{U}$ or $\overline{D}$, and will be clear from the context. We say that a set X is *cofinal* in $\widehat{s}$ (or *cofinal* above s) if and only if for each $t \in \widehat{s}$, there is some $u \in X$ such that $u \supseteq t$. A subset $L \subseteq D$ is called *large* if and only if there is some $s \in \mathbb{U}$ such that $\varphi^{-1}[L]$ is cofinal in $\widehat{s}$. We point out that since D is a set of coding nodes, for any $L \subseteq D$, $\varphi^{-1}[L]$ is a subset of $\mathbb{C}$.

Lemma 6.4. *Let $n < \omega$ and $L \subseteq D$ be given. Suppose $L = \bigcup_{i<n} L_i$ for some $L_i \subseteq D$. If L is large, then there is an $i < n$ such that L_i is large.*

Proof. Suppose not. Since L is large, there is some $t \in \mathbb{U}$ such that $\varphi^{-1}[L]$ is cofinal above t. Since L_0 is not large, there is some $s_0 \supseteq t$ such that $\varphi^{-1}[L_0] \cap \widehat{s_0} = \emptyset$. Given $i < n-1$ and s_i, since L_{i+1} is not large, there is some $s_{i+1} \supseteq s_i$ such that $\varphi^{-1}[L_{i+1}] \cap \widehat{s_{i+1}} = \emptyset$. At the end of this recursive construction, we obtain an $s_{n-1} \in \mathbb{S}$ such that for all $i < n$, $\varphi^{-1}[L_i] \cap \widehat{s_{n-1}} = \emptyset$. Hence, $\varphi^{-1}[L] \cap \widehat{s_{n-1}} = \emptyset$, contradicting that $\varphi^{-1}[L]$ is cofinal above t. □

Thus, any partition of a large set into finitely many pieces contains at least one piece which is large.

Given a subset $I \subseteq \omega$, let $\mathbf{K}\restriction I$ denote the substructure of $\mathbf{K}$ on vertices $\{v_i : i \in I\}$. Recalling that $\mathbf{J}$ denotes $\mathbf{K}\restriction D$, we let $\mathbf{J}\restriction I$ denote the substructure of $\mathbf{J}$ on vertices $\{v_i^D : i \in I\}$, where v_i^D is the vertex represented by the coding node c_i^D. The next lemma will be applied in two important ways. First, it will aid in finding splitting nodes in the meet-closure of D (denoted by $\mathrm{cl}(D)$) as needed to construct a similarity copy of a given antichain of coding nodes A inside D. Second, it will guarantee that we can find nodes in D which have the needed passing types in order to continue building a similarity copy of A in D.

Given a subset $L \subseteq \overline{D}$, we say that L is *large* exactly when $L \cap D$ is large. Note that since φ has range D, $\varphi^{-1}[L]$ is always a subset of $\mathbb{C}$. Given a finite set $I \subseteq \omega$ and 1-types σ, τ over $\mathbf{J}\restriction I$ and $\mathbf{K}\restriction I$, respectively, we write $\sigma \sim \tau$ exactly when for each $i \in I$, $\sigma(v_i^D; \mathbf{J}\restriction I_i) \sim \tau(v_i; \mathbf{K}\restriction I_i)$, where $I_i = \{j \in I : j < i\}$.

Lemma 6.5. *Suppose* t *is in* $\overline{D}$ *and* $\widehat{t}$ *is large. Let* $s_* \in \mathbb{U}$ *be such that* $\varphi^{-1}[\widehat{t}]$ *is cofinal in* $\widehat{s_*}$. *Let* i *be the index such that* $|s_*| = |c_i|$, *and let* $I \subseteq i$, $n \geq i$, $I' = I \cup \{n\}$, *and* $\ell = |c_n^D|$ *be given. For any complete* 1*-type* σ *over* $\mathbf{J} \restriction I'$ *such that* $\sigma \restriction (\mathbf{J} \restriction I) \sim s_* \restriction (\mathbf{K} \restriction I)$, *let*

$$L_\sigma = \bigcup \{\widehat{u} : u \in \widehat{t} \restriction \ell \text{ and } u \restriction (\mathbf{J} \restriction I') \sim \sigma\}. \tag{49}$$

Then L_σ *is large.*

Proof. Fix an $s \supseteq s_*$ with $|s| > |c_n|$ such that $s \restriction (\mathbf{K} \restriction I') \sim \sigma$ holds. Suppose towards a contradiction that L_σ is not large, and fix an extension $s' \supseteq s$ such that $\varphi^{-1}[L_\sigma] \cap \widehat{s'} = \emptyset$. Since $\varphi^{-1}[\widehat{t}]$ is cofinal in $\widehat{s_*}$, there is a coding node c_j in $\varphi^{-1}[\widehat{t}]$ extending s'. Notice that c_j being in $\varphi^{-1}[\widehat{t}]$ implies that $\varphi(c_j)$ extends t. Moreover, since c_j extends s and φ is passing type preserving, it follows that $\varphi(c_j) \restriction (\mathbf{J} \restriction I') \sim \sigma$. Thus, $\varphi(c_j)$ is in L_σ and hence, c_j is in $\varphi^{-1}[L_\sigma]$. But then $c_j \in \varphi^{-1}[L_\sigma] \cap \widehat{s'}$, a contradiction. □

For the remainder of the proof, fix a diagonal antichain of coding nodes $A \subseteq \mathbb{D}$. Let $\langle c_i^A : i < p \rangle$ enumerate the nodes in A in order of increasing length, where $p \leq \omega$, noting that each c_i^A is a coding node. For each $i < p$, let γ_i denote $\gamma_{c_i^A}$.

Let B denote the meet-closure of A; label the nodes of B as $\langle b_i : i < q \rangle$ in increasing order of length, where $q \leq \omega$. Thus, each node in B is either a member of A (hence, a coding node) or else a splitting node of degree two which is the meet of two nodes in A. Our goal is to build a similarity copy of B inside the meet-closure of D, denoted $\mathrm{cl}(D)$; that is, we aim to build a similarity map f from B into $\mathrm{cl}(D)$ so that $f[B] \sim B$. Now the map φ is already passing type preserving. The challenge is to get a $\prec$- and meet- preserving map which is still passing type preserving from B into $\mathrm{cl}(D)$.

First notice that $B \restriction 1 = A \restriction 1$. If we are working in $\mathbb{S}$, then $B \restriction 1$ is a subset of $D \restriction 1 = \mathbb{S}(0) = \Gamma$ with possibly more than one node. If we are working in $\mathbb{U}$, then $B \restriction 1$ is the singleton $D \restriction 1$. Without loss of generality, we may assume that $|c_0^A| > 1$. Let f_{-1} be the empty map, let T_{-1} denote $B \restriction 1$, let $N_{-1} = 1$, and let ψ_{-1} be the identity map on T_{-1}. Let $\widehat{D}$ be the tree induced by $\mathrm{cl}(D)$. Let $M_{-1} = 1$, and for each $k < q$, let $M_k = |b_{k-1}| + 1$, where we make the convention $|b_{-1}| = 0$.

For each $k < q$ we will recursively define meet-closed sets $T_k \subseteq \widehat{D}$, maps f_k and ψ_k, and $N_k < \omega$ such that the following hold:

(1) f_k is a +similarity embedding of $\{b_i : i < k\}$ into T_k.
(2) $|t| \leq N_k$ for all $t \in T_k$.
(3) All maximal nodes of T_k are either in $T_k \restriction N_k$, or else in the range of f_k.
(4) θ_k is a $\prec$ and passing type preserving bijection of $B \restriction M_k$ to $T_k \restriction N_k$.
(5) $T_{k-1} \subseteq T_k$, $f_{k-1} \subseteq f_k$, and $N_{k-1} < N_k$.

The idea behind T_k is that it will contain a similarity image of $\{b_i : i < k\} \cup (B \restriction M_k)$, the nodes in the image of $B \restriction M_k$ being the ones we need to continue extending in order to build a similarity copy of B in $\mathrm{cl}(D)$ If $p > 1$ in the Extension Property, then in (1) we further assume that f_k preserves the ψ value of the splitting node.

Assume now that $k < q$, and (1)–(6) hold for all $k' < k$. We have two cases.

Case I. b_k is a splitting node.

Let $i < j < p$ be such that $b_k = c_i^A \wedge c_j^A$. Let $t_k = \theta_k(b_k \restriction M_k)$, recalling that by (5), t_k is a member of $T_k \restriction N_k$. By (3), $\widehat{t_k}$ is large, so we can fix a coding node $c_n \in \varphi^{-1}[\widehat{t_k}]$. Then $c_n^D = \varphi(c_n) \supseteq t_k$. Let $N_{k+1} = |c_{n+1}^D|$.

Our goal is to find two incomparable nodes which extend t_k and have cones which are large. Recalling that $N_k = |t_k|$, let

$$I = \{i < \omega : |c_i^D| < N_k\}, \tag{50}$$

and let $I' = I \cup \{n\}$. Let σ and τ be distinct 1-types over $\mathbf{J}(I')$ such that both $\sigma \restriction \mathbf{J}(I)$ and $\tau \restriction \mathbf{J}(I)$ equal $t_k \restriction \mathbf{J}(I)$. For each $\mu \in \{\sigma, \tau\}$, let

$$L_\mu = \bigcup \{\widehat{u} : u \in \widehat{t_k} \restriction N_{k+1} \text{ and } u(c_n^D; \mathbf{J}(I)) = \mu\}. \tag{51}$$

By Lemma 6.5, both L_σ and L_τ are large. It then follows from Lemma 6.4 that there are $t_\sigma, t_\tau \in \widehat{t_k} \restriction N_{k+1}$ such that $t_\sigma \in L_\sigma$ and $t_\tau \in L_\tau$, and both $\widehat{t_\sigma}$ and $\widehat{t_\tau}$ are large. Since $\sigma \neq \tau$, it follows that $t_\sigma \neq t_\tau$. Hence, t_σ and t_τ are incomparable, since they have the same length, N_{k+1}. Since both $t_\sigma \supseteq t_k$ and $t_\tau \supseteq t_k$, we have $t_\sigma \wedge t_\tau \supseteq t_k$.

As $(B \cap \widehat{b_k}) \restriction M_k$ has size exactly two, define θ_{k+1} on $(B \cap \widehat{b_k}) \restriction M_{k+1}$ to be the unique $\prec$-preserving map onto $\{t_\sigma, t_\tau\}$. Let E_k denote $(B \setminus \widehat{b_k}) \restriction M_k$. For $s \in E_k$, choose some $t_s \in \widehat{\theta_k(s)} \restriction N_{k+1}$ such that $\widehat{t_s}$ is large. This is possible by Lemma 6.4, since $\bigcup\{\widehat{t} : t \in \widehat{\theta_k(s)} \restriction N_{k+1}\}$ is large. Every $s \in E_k$ has a unique extension $s' \in B \restriction M_{k+1}$. Define $\theta_{k+1}(s') = t_s$. Let f_{k+1} be the extension of f_k which sends b_k to $t_\sigma \wedge t_\tau$, and let

$$T_{k+1} = T_k \cup \{t_\sigma, t_\tau, t_\sigma \wedge t_\tau\} \cup \{t_s : s \in E_k\}. \tag{52}$$

For $\mathcal{COE}_{m,p}$, if $\psi(b_k) = m$, then we require σ and τ to both include $\neg E_{m-1}(x, v_n^D)$; if $\psi(b_k) < m$, then we require σ and τ to both include $E_{\psi(b_k)}(x, v_n^D)$ and $\neg E_i(x, v_n^D)$ for all $i < \psi(b_k)$. In general, if $\mathbf{K}$ satisfies (2) of the Extension Property, and if $\psi(b_k) = m$, then we take σ and τ above so that the pair $\{\sigma \restriction (\mathbf{K} \restriction \{v_n^D\}), \tau \restriction (\mathbf{K} \restriction \{v_n^D\})\}$ is in Q_m in (2b) of the Extension Property.

This completes Case I.

Case II. b_k is a coding node.

In this case, $b_k = c_j^A$ for some $j < p$. By the Induction Hypothesis, for each $t \in T_k \restriction N_k$, $\widehat{t}$ is large; so we can choose some $s_t \in \mathbb{U}$ such that $\varphi^{-1}[\widehat{t}]$ is cofinal above s_t. Fix $t_* = \theta_k(b_k \restriction M_k) \in T_k \restriction N_k$. Choose a coding node $c_n \supseteq s_{t_*}$ in $\mathbb{U}$ such that $|c_n| > \max\{|s_t| : t \in T_k \restriction N_k\}$ and $\gamma_{c_n} = \gamma_j$, the $\gamma \in \Gamma$ which the vertex v_j^A satisfies. (In the case that $\mathbb{D} \subseteq \mathbb{S}$, this γ_j is already guaranteed since $c_n \supseteq s_{t_*} \supseteq \gamma_j$. If $\mathbb{D} \subseteq \mathbb{U}$, there are cofinally many coding nodes extending s_{t_*} which satisfy γ_j.) Let d_k denote $c_n^D = \varphi(c_n)$, noting that $\gamma_{d_k} = \gamma_j$. Extend f_k by defining $f_{k+1}(b_k) = d_k$, and let $N_{k+1} = |c_{n+1}^D|$. If $q < \omega$ and $k = q - 1$, we are done. Otherwise, we must extend the other members of $(T_k \restriction N_k) \setminus \{t_*\}$ to nodes in $\widehat{D} \restriction N_{k+1}$ so as to satisfy (1)–(6).

For each $i \in \{k, k+1\}$, let $E_i = (B \restriction M_i) \setminus \{b_i \restriction M_i\}$. Fix an $s \in E_k$ and let $t = \theta_k(s)$, which is a node in $T_k \restriction N_k$. Note that there is a unique $s' \in E_{k+1}$ such that $s' \supseteq s$. Let $A{\downarrow}j$ denote $\{c_i^A : i \leq j\}$, σ denote $s' \restriction (A{\downarrow}j)$, and $f_k[A{\downarrow}j]$ denote $\{f_k(c_i^A) : i < j\}$. Let $I = \{i < \omega : c_i^D \in f_k[A{\downarrow}j]\}$. Our goal is to find a $t' \supseteq t$ with $|t'| > |d_k|$ such that $t'(d_k; \mathbf{J}(I)) \sim \sigma$.

Take c_m to be any coding node in $\mathbb{S}$ extending s such that $|c_m| > |c_n|$ and $c_m(c_n; A{\downarrow}j) \sim \sigma$. Such a c_m exists by SDAP. Then $\varphi(c_m)(d_k; \mathbf{J}(I)) \sim \sigma$, since φ is passing type preserving. By Lemma 6.5,

$$L_\sigma := \bigcup\{\widehat{u} : u \in \widehat{t}\restriction N_{k+1} \text{ and } u(d_k; f_k[A{\downarrow}j]) \sim \sigma\} \tag{53}$$

is large. Thus, by Lemma 6.4, there is some $u_s \in \widehat{t}\restriction N_{k+1}$ such that $\widehat{u}_s$ is large. Define $\theta_{k+1}(s) = u_s$. This builds

$$T_{k+1} = T_k \cup \{d_k\} \cup \{\psi_{k+1}(s) : s \in E_k\} \tag{54}$$

and concludes the construction in Case II.

Finally, let $f = \bigcup_k f_k$. Then f is a similarity map from B to $f[B]$, and thus, the antichain of coding nodes in $f[A]$ is similar to A. Therefore, all similarity types of diagonal antichains of coding nodes persist in $\mathbf{J}$. □

As the antichain in the previous theorem can be infinite, we immediately obtain the following corollary.

Corollary 6.6. *Suppose $\mathbf{K}$ satisfies SDAP$^+$. Given D a subset of $\mathbb{D}$ which represents a copy of $\mathbf{K}$, there is a subset D' of D such that $D' \sim \mathbb{D}$, assuming $\mathbf{K}$ satisfies SDAP$^+$.*

Combining the previous results, we obtain canonical partitions for Fraïssé classes that have Fraïssé limits satisfying SDAP$^+$; these canonical partitions are simply described by similarity types.

Theorem 6.7 (Simply characterized big Ramsey degrees)**.** *Let $\mathbf{K}$ be an enumerated Fraïssé structure for a Fraïssé class $\mathcal{K}$ such that $\mathbf{K}$ satisfies SDAP$^+$. Given $\mathbf{G} \in \mathcal{K}$, the partition $\{\mathrm{Sim}(C) : C \in \mathrm{Sim}(\mathbf{G})\}$ is a canonical partition of the copies of $\mathbf{G}$ in $\mathbf{K}$. It follows that the big Ramsey degree $T(\mathbf{G}, \mathbf{K})$ equals the number of similarity types of antichains of coding nodes in $\mathbb{T}$ representing $\mathbf{G}$. That is,*

$$T(\mathbf{G}, \mathbf{K}) = |\,\mathrm{Sim}(\mathbf{G})|.$$

Proof. Let $\mathbf{G} \in \mathcal{K}$ be given, and suppose h is a coloring of all copies of $\mathbf{G}$ in $\mathbf{K}$ into finitely many colors. By Theorem 5.22, there is an antichain of coding nodes $\mathbb{D} \subseteq \mathbb{T}$ which codes a copy of $\mathbf{K}$, and moreover, for each $C \in \mathrm{Sim}(\mathbf{G})$, h is constant on $\mathrm{Sim}_{\mathbb{D}}(C)$. Let $\mathbf{J} = \mathbf{K}\restriction\mathbb{D}$.

Given any subcopy $\mathbf{J}'$ of $\mathbf{J}$, Theorem 6.2 implies that $\mathrm{Sim}_D(C) \neq \emptyset$ for each $C \in \mathrm{Sim}(\mathbf{G})$, where $D = \mathbb{S}\restriction\mathbf{J}'$. Thus, $\{\mathrm{Sim}(C) : C \in \mathrm{Sim}(\mathbf{G})\}$ is a canonical partition of the copies of $\mathbf{G}$ in $\mathbf{K}$. It follows that $T(\mathbf{G}, \mathcal{K}) = |\,\mathrm{Sim}(\mathbf{G})|$. □

We now apply Theorem 6.7 to show that Fraïssé structures with SDAP$^+$ satisfy the conditions of Zucker's Theorem 7.1 in [46], yielding the Main Theorem. Zucker used colorings of embeddings rather than colorings of copies throughout [46]. Our task now is to translate Theorem 6.7, which uses colorings of copies of a given structure, into the setting of [46]. To do so, we need to review the following notions from [46].

Let $\mathbf{K}$ be an enumerated Fraïssé structure for a Fraïssé class $\mathcal{K}$. An *exhaustion* of $\mathbf{K}$ is a sequence $\{\mathbf{A}_n : n < \omega\}$ with each $\mathbf{A}_n \in \mathcal{K}$, $\mathbf{A}_n \subseteq \mathbf{A}_{n+1} \subseteq \mathbf{K}$, such that $\mathbf{K} = \bigcup_{n<\omega} \mathbf{A}_n$. Given $m \le n$, write $H_m := \mathrm{Emb}(\mathbf{A}_m, \mathbf{K})$ and $H^n_m := \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$. For $f \in H^n_m$, the function $\hat{f} : H_n \to H_m$ is defined by $\hat{f}(s) = s \circ f$, for each $s \in H_n$.

(Here we are using Zucker's notation, so s is denoting an embedding rather than a node in $\mathbb{U}$.)

The following terminology is found in Definition 4.2 in [46]. A set $S \subseteq H_m$ is *unavoidable* if for each embedding $\eta : \mathbf{K} \to \mathbf{K}$, we have $\eta^{-1}(S) \neq \emptyset$. Fix $k \leq r < \omega$ and let $\gamma : H_m \to r$ be a coloring. We call γ an *unavoidable k-coloring* if the image of γ, written $\mathrm{Im}(\gamma)$, has cardinality k, and for each $i < r$, we have $\gamma^{-1}(\{i\}) \subseteq H_m$ is either empty or unavoidable. Thus, an unavoidable coloring is essentially the same concept as persistence, with the addition that attention is also given to the embedding.

The following is taken from Definition 4.7 in [46]: Let γ and δ be colorings of H_m. We say that δ *refines* γ and write $\gamma \leq \delta$ if whenever $f_0, f_1 \in H_m$ and $\delta(f_0) = \delta(f_1)$, then $\gamma(f_0) = \gamma(f_1)$. For $m \leq n < \omega$, γ a coloring of H_m, and δ a coloring of H_n, we say that δ *strongly refines* γ and write $\gamma \ll \delta$ if for every $f \in H^n_m$, we have that $\gamma \circ \hat{f} \leq \delta$.

Theorem 7.1 in [46], which we state next, provides conditions for showing that a Fraïssé limit admits a big Ramsey structure. We will then apply this theorem to show that all Fraïssé structures with SDAP$^+$ admit big Ramsey structures, thus giving the Main Theorem.

Theorem 6.8 (Zucker, [46])**.** *Let $\mathbf{K} = \bigcup_{n<\omega} \mathbf{A}_n$ be a Fraïssé structure, where $\{\mathbf{A}_n : n < \omega\}$ is an exhaustion of $\mathbf{K}$, and suppose each $\mathbf{A}_n$ has finite big Ramsey degree R_n in $\mathbf{K}$. Assume that for each $m < \omega$, there is an unavoidable R_m-coloring γ_m of H_m so that $\gamma_m \ll \gamma_n$ for each $m \leq n < \omega$. Then $\mathbf{K}$ admits a big Ramsey structure.*

Now we show how to translate our results so as to apply Theorem 6.8. Given an enumerated Fraïssé structure $\mathbf{K}$, we point out that $\{\mathbf{K}_n : n < \omega\}$ is an exhaustion of $\mathbf{K}$. Theorem 6.7 shows that $\mathbf{K}_n$ has finite big Ramsey degree $T(\mathbf{K}_n, \mathbf{K}) = |\mathrm{Sim}(\mathbf{K}_n)|$ for colorings of *copies* of $\mathbf{K}_n$ in $\mathbf{K}$. Recalling Remark 2.5, the big Ramsey degree for *embeddings* of $\mathbf{K}_n$ into $\mathbf{K}$ is $T(\mathbf{K}_n, \mathbf{K}) \cdot |\mathrm{Aut}(\mathbf{K}_n)|$.

Theorem 6.9. *Suppose $\mathcal{K}$ is a Fraïssé class with Fraïssé limit $\mathbf{K}$ and with canonical partitions characterized via diagonal antichains of coding nodes in a coding tree of 1-types. Then the conditions of Theorem 6.8 are satisfied.*

Proof. Recalling that $\mathbb{D}$ denotes the diagonal antichain of coding nodes constructed in Lemma 5.20, we shall abuse notation and use $\mathbf{K}$ to denote the structure $\mathbf{K} \restriction \mathbb{D}$. Thus, the universe of $\mathbf{K}$ will (without loss of generality) be ω, and embeddings s of initial segments $\mathbf{K}_n$ into $\mathbf{K}$ will produce diagonal antichains $\mathbb{D} \restriction s[\mathbf{K}_n] \subseteq \mathbb{D}$. Given $n < \omega$, let $T_n := T(\mathbf{K}_n, \mathbf{K})$, and let $\langle C^n_0, \dots, C^n_{T_n-1}\rangle$ be an enumeration of $\mathrm{Sim}(\mathbf{K}_n)$, a set of representatives of the similarity types of diagonal antichains of coding nodes representing a copy of $\mathbf{K}_n$. Let $\mathrm{Aut}(\mathbf{K}_n)$ denote the set of automorphisms of $\mathbf{K}_n$.

As $\mathbf{K}_n$ has vertex set $n = \{0, \dots, n-1\}$, its vertex set is linearly ordered. Given $s \in H_n$, let $\mathbf{A} := s[\mathbf{K}_n]$, with vertex set $\langle \mathrm{a}_0, \dots, \mathrm{a}_{n-1}\rangle$ written in increasing order as a subset of ω. Let p_s denote the permutation of n defined by $s(j) = \mathrm{a}_{p_s(j)}$, for $j < n$. Given $\ell < T_n$, let $\mathbf{C}^n_\ell$ denote the structure $\mathbf{K} \restriction C^n_\ell$, and let $\langle v^\ell_0, \dots, v^\ell_{n-1}\rangle$ denote the vertex set of $\mathbf{C}^n_\ell$ in increasing order as a subset of ω. Let P_ℓ be the set of permutations p of n such that the map $j \mapsto v^\ell_{p(j)}$, $j < n$, induces an isomorphism from $\mathbf{K}_n$ to $\mathbf{C}^n_\ell$. Note that $|P_\ell| = |\mathrm{Aut}(\mathbf{K}_n)|$.

Letting $R_n = T(\mathbf{K}_n, \mathbf{K}) \cdot |\mathrm{Aut}(\mathbf{K}_n)|$, we define an unavoidable coloring $\gamma_n : H_n \to R_n$ as follows: For $s \in H_n$, define $\gamma_n(s) = \langle t, p_s \rangle$, where $t < T_n$ is the index satisfying $\mathbb{D} \restriction \mathbf{B}_s \sim C^n_t$. Then γ_n is an unavoidable coloring, by Theorem 6.2.

Let $m \le n < \omega$. To show that $\gamma_m \ll \gamma_n$, we start by fixing $f \in H^n_m$ and $s, t \in H_n$ such that $\gamma_n(s) = \gamma_n(t)$. Note that $f : \mathbf{K}_m \to \mathbf{K}_n$ is completely determined by its behavior on the sets of vertices. Thus, we equate f with its induced injection from m into n. Let $\mathbf{A}, \mathbf{B}$ denote the structures $s[\mathbf{K}_n], t[\mathbf{K}_n]$, respectively. Let $A = \mathbb{D} \restriction \mathbf{A}$ and $B = \mathbb{D} \restriction \mathbf{B}$, the diagonal antichains of coding nodes representing the structures $\mathbf{A}, \mathbf{B}$, respectively. Since $\gamma_n(s) = \gamma_n(t)$, it follows that $A \sim B$ and $p_s = p_t$. It follows that $p_s \circ f = p_t \circ f$.

Our task is to show that $\gamma_m(\hat{f}(s)) = \gamma_m(\hat{f}(t))$. Letting $\langle \mathrm{a}_0, \dots, \mathrm{a}_{n-1} \rangle$ denote the increasing enumeration of the vertices in $\mathbf{A}$, we see that $s \circ f$ is an injection from m into $\{\mathrm{a}_j : j < n\}$. Letting $\bar{m} = \{j < n : \exists i < m\, (\mathrm{a}_j = s \circ f(i))\}$, and letting μ be the strictly increasing injection from $\bar{m}$ into m, we see that $p_{\hat{f}(s)}$ is the permutation of m given by $p_{\hat{f}(s)}(i) = \mu \circ f \circ p_s(i)$. Likewise, $t \circ f$ is an injection from m into $\{\mathrm{b}_j : j < n\}$, where $\langle \mathrm{b}_0, \dots, \mathrm{b}_{n-1} \rangle$ denotes the increasing enumeration of the vertices in $\mathbf{B}$. Since $p_s = p_t$, we see that $f \circ p_s = f \circ p_t$, and hence, the set of indices $\{j < n : \exists i < m\, (\mathrm{b}_j = t \circ f(i))\}$ equals $\bar{m}$. Thus, $p_{\hat{f}(t)}(i) = \mu \circ f \circ p_t(i)$ for each $i < m$. Hence, $p_{\hat{f}(s)} = p_{\hat{f}(t)}$.

$\hat{f} \circ s$ maps $\mathbf{K}_m$ to the substructure $\mathbf{A}'$ of $\mathbf{A}$ on vertices $\{\mathrm{a}_{p_s \circ f(i)} : i < m\}$. This substructure induces the antichain of coding nodes $A' := \{c^A_{p_s \circ f(i)} : i < m\} \subseteq A$; that is, $A' = A \restriction \mathbf{A}'$. Similarly, $t \circ \hat{f}$ maps $\mathbf{K}_m$ to the substructure $\mathbf{B}'$ of $\mathbf{B}$ on vertices $\{\mathrm{b}_{p_t \circ f(i)} : i < m\}$; this induces the antichain of coding nodes $B' := B \restriction \mathbf{B}' = \{c^B_{p_s \circ f(i)} : i < m\} \subseteq B$. Since $p_s = p_t$, we have $p_s \circ f = p_t \circ f$, and since $A \sim B$, it follows that $A' \sim B'$. Let $\ell < T_m$ be the index such that $A' \sim B' \sim C^m_\ell$. Then $\gamma_m(\hat{f}(s)) = (\ell, p_{\hat{f}(s)}) = \gamma_m(\hat{f}(t))$, since $p_{\hat{f}(s)} = p_{\hat{f}(t)}$. Therefore, $\gamma_m \ll \gamma_n$. □

The big Ramsey structure of a Fraïssé limit $\mathbf{K}$ with SDAP$^+$ is obtained simply by expanding the language $\mathcal{L}$ of $\mathbf{K}$ to the language $\mathcal{L}^* = \mathcal{L} \cup \{\triangleleft, \mathscr{Q}\}$, where $\triangleleft$ and $\mathscr{Q}$ are not in $\mathcal{L}$, $\triangleleft$ is a binary relation symbol, and $\mathscr{Q}$ is a quaternary relation symbol. In fact, by Theorem 6.9, this will be the case for any Fraïssé class with canonical partitions characterized via diagonal antichains of coding nodes in a coding tree of 1-types. The big Ramsey $\mathcal{L}^*$-structure $\mathbf{K}^*$ for $\mathbf{K}$ is described as follows.

Let $\mathbb{D}$ be the diagonal antichain of coding nodes from the proof of Theorem 6.9, and recall the linear order $\prec$ on $\mathbb{S}$ described in Subsection 4.2 (see paragraph following Fact 4.14). Note that $(\mathbb{D}, \prec)$ is isomorphic to the rationals as a linear order. Following Zucker in Section 6 of [46], let R be the quaternary relation on $\mathbb{D}$ given by: For $p \preccurlyeq q \preccurlyeq r \preccurlyeq s \in \mathbb{D}$, set

$$R(p, q, r, s) \iff |p \wedge q| \le |r \wedge s|, \tag{55}$$

where $p \preccurlyeq q$ means either $p \prec q$ or $p = q$. Without loss of generality, we may use $\mathbf{K}$ to denote $\mathbf{K} \restriction \mathbb{D}$. Define $\mathbf{K}^*$ be the expansion of $\mathbf{K}$ to the language $\mathcal{L}^*$ in which $\triangleleft$ is interpreted as $\prec$ and $\mathscr{Q}$ is interpreted as R. Then we have the following.

Theorem 6.10. *Let $\mathcal{K}$ be a Fraïssé class in language $\mathcal{L}$ and $\mathbf{K}$ a Fraïssé limit of $\mathcal{K}$ Suppose that $\mathbf{K}$ satisfies SDAP$^+$, and let $\mathcal{L}^* = \mathcal{L} \cup \{\triangleleft, \mathscr{Q}\}$, where $\triangleleft$ is a binary relation symbol and $\mathscr{Q}$ is a quaternary relation symbol. Then the $\mathcal{L}^*$-structure $\mathbf{K}^*$ is a big Ramsey structure for $\mathbf{K}$.*

Proof. Theorem 6.9 implies the existence of a big Ramsey structure for $\mathbf{K}$. Moreover, the proof of Theorem 6.9 shows that $\mathbf{K}^*$ satisfies Definition 2.6 of a big Ramsey structure. □

7. SDAP$^+$ implies the Ordered Ramsey Property

From the results in the previous two sections, we can quickly deduce Theorem 7.2 below: The ordered expansion of the age of any Fraïssé structure satisfying SDAP$^+$ is a Ramsey class. This theorem offers a new approach for proving that a Fraïssé class has an ordered expansion which is Ramsey, complementing, in the case of classes whose Fraïssé limits have SDAP$^+$, the famous partite construction method of Nešetřil and Rödl (see [36] and [37]) which is at the heart of finite structural Ramsey theory.

Given a Fraïssé class $\mathcal{K}$ in a finite relational language $\mathcal{L}$, let $<$ be an additional binary relation symbol not in $\mathcal{L}$, and let $\mathcal{L}' = \mathcal{L} \cup \{<\}$. Let $\mathcal{K}^<$ denote the class of all ordered expansions of structures in $\mathcal{K}$, namely, the collection of all $\mathcal{L}'$-structures in which $<$ is interpreted as a linear order and whose reducts to the language $\mathcal{L}$ are members of $\mathcal{K}$. Since $\mathcal{K}$ has disjoint amalgamation by assumption, $\mathcal{K}^<$ will be a Fraïssé class. We denote the Fraïssé limit of $\mathcal{K}^<$ by $\mathbf{K}^<$, and note that $\mathbf{K}^<$ is universal for all countable $\mathcal{L}'$-structures in which the relation symbol $<$ is interpreted as a linear order. We shall write $\mathbf{M}' := \langle \mathbf{M}, <' \rangle$ for any $\mathcal{L}'$-structure interpreting $<$ as a linear order; it will be understood that $\mathbf{M}$ is an $\mathcal{L}$-structure and that $<'$ is the linear order on M interpreting $<$.

Definition 7.1. Given a Fraïssé class $\mathcal{K}$ and an enumerated Fraïssé structure $\mathbf{K}$, let $\mathbb{U}$ be the unary-colored coding tree of 1-types for $\mathbf{K}$. We call a finite antichain C of coding nodes in $\mathbb{U}$ a *comb* if and only if for any two coding nodes c, c' in C,

$$|c| < |c'| \iff c \prec c', \tag{56}$$

where $\prec$ is the lexicographic order on T.

Theorem 7.2. *Let $\mathcal{K}$ be a Fraïssé class in a finite relational language $\mathcal{L}$, and suppose that the Fraïssé limit of $\mathcal{K}$ has SDAP$^+$. Then the ordered expansion $\mathcal{K}^<$ of $\mathcal{K}$ has the Ramsey property.*

Proof. Let $\mathbf{K}$ be any enumerated Fraïssé limit of $\mathcal{K}$. Then $\mathbf{K}$ has universe ω, and may be regarded as a linearly ordered structure in order-type ω, that is, as an $\mathcal{L}'$-structure $\langle \mathbf{K}, \in \rangle$ in which the relation symbol $<$ is interpreted as the order inherited from ω. Let $\mathbb{U}$ be the coding tree of 1-types associated with $\mathbf{K}$.

Let $\mathbf{A}', \mathbf{B}'$ be members of $\mathcal{K}^<$ such that $\mathbf{A}'$ embeds into $\mathbf{B}'$. Fix a finite coloring f of all copies of $\mathbf{A}'$ in $\langle \mathbf{K}, \in \rangle$. Note that in this context, a substructure $\langle \mathbf{A}^*, \in \rangle$ of $\langle \mathbf{K}, \in \rangle$ is a copy of $\mathbf{A}'$ when there is an $\mathcal{L}'$-isomorphism between $\langle \mathbf{A}, <' \rangle$ and $\langle \mathbf{A}^*, \in \rangle$.

Let $\mathbb{D}$ be the antichain of coding nodes from Lemma 5.20 representing a copy of $\mathbf{K}$, and let $A \subseteq \mathbb{D}$ be a comb representing $\mathbf{A}'$. Thus, if $\langle c_i^A : i < m \rangle$ is the enumeration of A in order of increasing length, then the coding node c_i^A represents the i-th vertex of $\mathbf{A}'$ (according to its linear ordering $<'$).

Let f^* be the coloring on $\mathrm{Sim}(A)$ induced by f. By Theorem 6.2 there is a substructure $\mathbf{J}$ of $\mathbf{K} \restriction \mathbb{D}$ such that $\mathbf{J} \cong^\omega \mathbf{K}$ and every similarity copy of A in $D := \mathbb{U} \restriction \mathbf{J}$ has the same f^* color. Again by Theorem 6.2, there is a subset $B^* \subseteq D$ such that B^* is a comb representing a copy of $\mathbf{B}'$ in the order inherited on the coding nodes in B^*. Then every copy of $\mathbf{A}'$ represented by a set of coding nodes in B^* is represented by a comb, and hence has the same f-color. Since $\langle \mathbf{K}, \in \rangle$ is an $\mathcal{L}'$-structure interpreting the relation symbol $<$ as a linear order, $\langle \mathbf{K}, \in \rangle$ embeds into the Fraïssé limit of $\mathcal{K}^<$, and so it follows from Definition 2.1 that $\mathcal{K}^<$ has the Ramsey property. □

Remark 7.3. It is impossible for any comb to represent a copy of a Fraïssé structure $\mathbf{K}$ satisfying SDAP$^+$ when $\mathbf{K}$ has at least one non-trivial relation of arity at least two. The contrast between similarity types of diagonal antichains of 1-types persisting in every copy of $\mathbf{K}$ in a coding tree and combs (or any other fixed similarity type) being sufficient to prove the Ramsey property for the ordered expansion of its age lies at the heart of the difference between big Ramsey degrees for $\mathbf{K}$ and the Ramsey property for $\mathcal{K}^<$.

In the paper [23], Hubička and Nešetřil prove general theorems from which the majority of Ramsey classes can be deduced. In particular, Corollary 4.2 of [23] implies that every relational Fraïssé class with free amalgamation has an ordered expansion with the Ramsey property. So for Fraïssé classes satisfying SFAP, Theorem 7.2 provides a new proof of special case of a known result. However, we are not aware of a prior result implying Theorem 7.2 in its full generality.

A different approach to recovering the ordered Ramsey property is given in [22]. In that paper, Hubička's results on big Ramsey degrees via the Ramsey theory of parameter spaces recover the Nešetřil-Rödl theorem [36] that the class of finite ordered triangle-free graphs has the Ramsey property.

These approaches to proving the Ramsey property for ordered Fraïssé classes may seem at first glance very different from the partite construction method. However, the methods must be related at some fundamental level, similarly to the relationship between the Halpern-Läuchli and Hales-Jewett theorems. It will be interesting to see if this could lead to new Hales-Jewett theorems corresponding to the various forcing constructions (in [13], [11], [46], and this paper) which have been used to determine finite and exact big Ramsey degrees.

8. Concluding remarks and open problems

In Section 3, we gave examples of Fraïssé classes with Fraïssé limits satisfying SDAP$^+$. By the Main Theorem, any such Fraïssé limit has finite big Ramsey degrees and admits a big Ramsey structure that has a simple characterization.

Question 8.1. Which other Fraïssé classes either satisfy SFAP, or more generally, have Fraïssé limits satisfying SDAP$^+$?

Fraïssé structures consisting of finitely many independent linear orders present an interesting case as we do not know whether they satisfy the Diagonal Coding Property, and hence whether they have SDAP$^+$, but their ages do have SDAP, and their coding trees have bounded branching. This motivates the formulation of the following properties: For $k \geq 2$, we say that the Fraïssé limit $\mathbf{K}$ of a Fraïssé class $\mathcal{K}$ satisfies *k-SDAP$^+$* if $\mathbf{K}$ satisfies SDAP; there is a perfect subtree $\mathbb{T}$ of the coding tree of 1-types $\mathbb{U}$ for $\mathbf{K}$ such that $\mathbb{T}$ represents a copy of $\mathbf{K}$ and $\mathbb{T}$ has splitting nodes with degree $\leq k$; and the appropriately formulated Extension Property holds. Note that here, we are not requiring $\mathbb{T}$ to be a skew tree: $\mathbb{T}$ is allowed to have more than one splitting node on any given level. We let BSDAP$^+$ stand for *Bounded SDAP$^+$*, meaning that there is a $k \geq 2$ such that k-SDAP$^+$ holds. This brings us to the following implications.

Fact 8.2. *SFAP $\Longrightarrow$ SDAP$^+$ $\Longrightarrow$ 2-SDAP$^+$ $\Longrightarrow$ BSDAP$^+$ $\Longrightarrow$ SDAP.*

Theorem 4.28 showed that SFAP implies SDAP$^+$. By definition, SDAP$^+$ implies 2-SDAP$^+$, which in turn implies BSDAP$^+$. Each of these properties implies SDAP, again by definition. The example of finitely many independent linear orders shows that BSDAP$^+$ does not imply SDAP$^+$. On the other hand, all examples considered in this paper satisfying SDAP also satisfy BSDAP$^+$. It could well be the case that SDAP is equivalent to BSDAP$^+$. The methods in this paper can be adjusted to handle structures with BSDAP$^+$, so the following question becomes interesting.

Question 8.3. Are SDAP, BSDAP$^+$, and 2-SDAP$^+$ equivalent? In other words, does SDAP imply BSDAP$^+$, and does BSDAP$^+$ imply 2-SDAP$^+$?

In Theorem 3.6, we proved that the Fraïssé limit of any free amalgamation class with forbidden 3-irreducible substructures admits big Ramsey structures. We also pointed out that the pyramid-free 3-hypergraphs do not fall under the purview of Theorem 3.6, as a pyramid is irreducible but not 3-irreducible. (Recall that by "pyramid" we mean a 3-hypergraph on four vertices with exactly three 3-hyperedges.) By the Nešetřil-Rödl Theorem [36], the Fraïssé class of ordered pyramid-free 3-hypergraphs has the Ramsey property. Combining methods in this paper with methods applicable to constrained free amalgamation classes of binary relational structures could be a strategy to yield finite big Ramsey degrees. More generally, we ask the following question, which would finish the main work on big Ramsey degrees of relational free amalgamation classes.

Question 8.4. Let $\mathcal{L}$ be a finite relational language and $\mathcal{F}$ a finite collection of finite irreducible $\mathcal{L}$-structures such that at least one structure in $\mathcal{F}$ is not 3-irreducible. Write Forb($\mathcal{F}$) for the free amalgamation class of finite $\mathcal{L}$-structures into which no member of $\mathcal{F}$ embeds. Does the Fraïssé limit of Forb($\mathcal{F}$) have finite big Ramsey degrees? If so, characterize the degrees to show that the Fraïssé limit of Forb($\mathcal{F}$) admits a big Ramsey structure.

Throughout this paper, we have mentioned known results regarding finite big Ramsey degrees. Actual calculations of big Ramsey degrees, however, are still sparse, and have only been found for the rationals by Devlin in [9], the Rado graph by Larson in [31], the structures $\mathbb{Q}_n$ and $\mathbf{S}(2)$ by Laflamme, Nguyen Van Thé, and Sauer in [29], and the rest of the circular digraphs $\mathbf{S}(n)$, $n \geq 3$, by Barbosa in [5]. The canonical partitions in Theorem 6.7 provide a template for calculating the big Ramsey degrees for all Fraïssé structures satisfying SDAP$^+$.

Problem 8.5. Calculate the big Ramsey degrees $T(\mathbf{A}, \mathbf{K})$, $\mathbf{A} \in \mathcal{K}$, for each Fraïssé class $\mathcal{K}$ with a Fraïssé limit satisfying SDAP$^+$.

Lastly, it is our hope that using combs in trees of 1-types might lead to smaller bounds for the ordered Ramsey property.

Problem 8.6. Suppose $\mathcal{K}$ is a Fraïssé class with Fraïssé limit satisfying SDAP$^+$. Use Theorem 7.2 to find better bounds for the smallest size of a structure $\mathbf{C} \in \mathcal{K}^<$ such that

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}} \tag{57}$$

for any given $\mathbf{A} \leq \mathbf{B}$ inside $\mathcal{K}^<$.

References

1. M. Balko, D. Chodounský, N. Dobrinen, J. Hubička, M. Konečný, L. Vena, and A. Zucker, *Exact big Ramsey degrees for binary relational structures with forbidden irreducible substructures*, (2020), In preparation.
2. M. Balko, D. Chodounský, J. Hubička, M. Konečný, and L. Vena, *Big Ramsey degrees of 3-uniform hypergraphs*, Acta Mathematica Universitatis Comenianae **LXXXVIII** (2019), no. 3, 415–422.
3. ———, *Big Ramsey degrees of 3-uniform hypergraphs are finite*, (2020), 9 pp, Submitted. arXiv:2008.00268.
4. ———, *Big Ramsey degrees of unconstrained relational structures*, (2020), In preparation.
5. Keegan Dasilva Barbosa, *A categorical notion of precompact expansions*, Archive for Mathematical Logic (2020), 29 pp, Submitted. arXiv:2002.11751.
6. Manuel Bodirsky, *Ramsey classes: Examples and constructions*, London Mathematical Society Lecture Note Series, 424, Cambridge University Press, 2015.
7. Peter Cameron, *Oligomorphic Permutation Groups*, Cambridge University Press, 1990.
8. Gabriel Conant, *An axiomatic approach to free amalgamation*, Journal of Symbolic Logic **82** (2017), no. 2, 648–671.
9. Dennis Devlin, *Some partition theorems for ultrafilters on ω*, Ph.D. thesis, Dartmouth College, 1979.
10. Natasha Dobrinen, *Borel sets of Rado graphs and Ramsey's theorem*, Proceedings of the 2016 Prague DocCourse on Ramsey Theory, 29 pp, To appear. arXiv:1904.00266v1.
11. ———, *Ramsey theory of the universal homogeneous k-clique-free graph*, 75 pp, Submitted. arXiv:1901.06660.
12. ———, *Ramsey theory of the universal homogeneous triangle-free graph, Part II: Exact big Ramsey degrees*, 22pp, arXiv:2009.01985.
13. ———, *The Ramsey theory of the universal homogeneous triangle-free graph*, Journal of Mathematical Logic **20** (2020), no. 2, 2050012, 75 pp.
14. ———, *Ramsey Theory on infinite structures and the method of strong coding trees*, Contemporary Logic and Computing (Adrian Rezus, ed.), College Publications, London, 2020, pp. 444–467.
15. Natasha Dobrinen, Claude Laflamme, and Norbert Sauer, *Rainbow Ramsey simple structures*, Discrete Mathematics **339** (2016), no. 11, 2848–2855.
16. Mohamed El-Zahar and Norbert Sauer, *The indivisibility of the homogeneous K_n-free graphs*, Journal of Combinatorial Theory, Series B **47** (1989), no. 2, 162–170.
17. ———, *On the divisibility of homogeneous hypergraphs*, Combinatorica **14** (1994), no. 2, 159–165.
18. Paul Erdős and Richard Rado, *A partition calculus in set theory*, Bulletin of the American Mathematical Society **62** (1956), 427–489.
19. Roland Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Annales Scientifiques de l'École Normale Supérieure **71** (1954).
20. Wilfred Hodges, *A Shorter Model Theory*, Cambridge University Press, 1997.
21. John Howe, *Big Ramsey degrees in homogeneous structures*, Ph.D. thesis, University of Leeds, Expected 2020.
22. Jan Hubička, *Big Ramsey degrees using parameter spaces*, (2020), 19 pp, Preprint. arXiv:2009.00967.
23. Jan Hubička and Jaroslav Nešetřil, *All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)*, Advances in Mathematics **356** (2019), 106791, 89 pp.
24. A. A. Ivanov, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, The Journal of Symbolic Logic **64** (1999), no. 2, 775–789.
25. Alexander Kechris, Vladimir Pestov, and Stevo Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geometric and Functional Analysis **15** (2005), no. 1, 106–189.
26. Alexander S. Kechris and Christian Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proceedings of the London Mathematical Society. Third Series **94** (2007), no. 2, 302–350.
27. Péter Komjáth and Vojtěch Rödl, *Coloring of universal graphs*, Graphs and Combinatorics **2** (1986), no. 1, 55–60.

28. Alex Kruckman, *Disjoint n-amalgamation and pseudofinite countably categorical theories*, Notre Dame Journal of Formal Logic **60** (2019), no. 1, 139–160.
29. Claude Laflamme, Lionel Nguyen Van Thé, and Norbert Sauer, *Partition properties of the dense local order and a colored version of Milliken's theorem*, Combinatorica **30** (2010), no. 1, 83–104.
30. Claude Laflamme, Norbert Sauer, and Vojkan Vuksanovic, *Canonical partitions of universal structures*, Combinatorica **26** (2006), no. 2, 183–205.
31. Jean Larson, *Counting canonical partitions in the random graph*, Combinatorica **28** (2008), no. 6, 659–678.
32. Richard Laver, unpublished.
33. Dragan Mašulović, *Finite big Ramsey degrees in universal structures*, Journal of Combinatorial Theory, Series A **170** (2020), 30 pp.
34. ———, *Ramsey degrees: big v. small*, (2020), 38 pp. arXiv:2003.03786v3.
35. Keith R. Milliken, *A Ramsey theorem for trees*, Journal of Combinatorial Theory, Series A **26** (1979), 215–237.
36. Jaroslav Nešetřil and Vojtěch Rödl, *Partitions of finite relational and set systems*, Journal of Combinatorial Theory Series A **22** (1977), no. 3, 289–312.
37. ———, *Ramsey classes of set systems*, Journal of Combinatorial Theory Series A **34** (1983), no. 2, 183–201.
38. Lionel Nguyen Van Thé, *Big Ramsey degrees and divisibility in classes of ultrametric spaces*, Canadian Mathematical Bulletin **51** (2008), no. 3, 413–423.
39. ———, *Structural Ramsey theory with the Kechris-Pestov-Todorcevic correspondence in mind*, Habilitation thesis, Université d'Aix-Marseille, 2013, p. 48 pp.
40. Frank P. Ramsey, *On a problem of formal logic*, Proceedings of the London Mathematical Society **30** (1929), 264–296.
41. Norbert Sauer, *Canonical vertex partitions*, Combinatorics, Probability, and Computing **12** (2003), no. 6, 671–704.
42. ———, *Coloring subgraphs of the Rado graph*, Combinatorica **26** (2006), no. 2, 231–253.
43. ———, *Vertex partitions of metric spaces with finite distance sets*, Discrete Mathematics **312** (2012), no. 1, 119–128.
44. Daoud Siniora and Sławomir Solecki, *Coherent extension of partial automorphisms, free amalgamation and automorphism groups*, Journal of Symbolic Logic **85** (2020), no. 1, 199–223.
45. Stevo Todorcevic, *Introduction to Ramsey Spaces*, Princeton University Press, 2010.
46. Andy Zucker, *Big Ramsey degrees and topological dynamics*, Groups, Geometry and Dynamics (2018), 235–276.
47. ———, *A Note on Big Ramsey degrees*, (2020), 21pp, Submitted. arXiv:2004.13162.

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