Sealing the Universally Baire sets

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We identify $\mathbb{R}$ with ω^ω equipped with the product topology. A *tree* T on $\omega \times \lambda$ is a subset of $\omega^{<\omega} \times \lambda^{<\omega}$ such that for all $(s, t) \in T$, $\mathrm{lh}(s) = \mathrm{lh}(t)$, and for all $i < n$, $(s{\restriction}i, t{\restriction}i) \in T$. We define the projection

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A set of reals A is γ-**universally Baire** if there are trees T, U on $\omega \times \lambda$ for some λ such that $A = p[T] = \mathbb{R} \backslash p[U]$ and whenever g is a $\leq \gamma$-generic, in $V[g]$, $p[T] = \mathbb{R} \backslash p[U]$.

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Theorem (Feng-Magidor-Woodin, 1992, [FMW92])

*A is **universally Baire** if and only if A is γ-universally Baire for all γ.*

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- Large cardinals were used to establish a plethora of results that generalize Shoenfield's Absoluteness Theorem to more complex formulas than Σ^1_2. UB sets are the largest class of sets for which a Shoenfield-type absoluteness can hold. If sufficient generic absoluteness is true about a set of reals then that set is universally Baire (the Tree Production Lemma). Roughly, let A_ϕ be the set of reals defined by ϕ. If sufficiently many statements about A_ϕ are generically absolute then it is because A_ϕ is universally Baire.

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$\forall \kappa \quad \forall^* X \prec H_\kappa \quad \forall g \in V$ generic over X & $y \in X[g] \cap \mathbb{R}$: $X[g] \models \phi[y] \iff V \models \phi[y]$

Sealing

Let Hom^∞ be the set of universally Baire sets. Given a generic g, we let $Hom^\infty_g = (Hom^\infty)^{V[g]}$ and $\mathbb{R}_g = \mathbb{R}^{V[g]}$. Also, if $A = p[T]$ for some tree T, then let $A^g = p[T] \cap V[g]$.

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Woodin showed that if A is a universally Baire set of reals and the universe has a class of Woodin cardinals then the theory of $L(A, \mathbb{R})$ cannot be changed. Thus, the next place to look for generic absoluteness is Hom^{∞}.

$L(A,\mathbb{R}) \vDash AD$

For any g, $\exists j: L(A,\mathbb{R}) \to L(A^g,\mathbb{R}_g)$

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AD

Definition (Woodin)

Sealing is the conjunction of the following statements.

1. For every set generic g, $L(Hom^\infty_g, \mathbb{R}_g) \models \mathrm{AD}^+$ and $\boxed{\mathcal{P}(\mathbb{R}_g) \cap L(Hom^\infty_g, \mathbb{R}_g) = Hom^\infty_g}$.
2. For every set generic g over V, for every set generic h over $V[g]$, there is an elementary embedding

$$j : L(Hom^\infty_g, \mathbb{R}_g) \to L(Hom^\infty_h, \mathbb{R}_h).$$

such that for every $A \in Hom^\infty_g$, $j(A) = A^h$.

Sealing dichotomy

If $\mathcal{M}$ is a model that conforms to the norms of modern inner model theory and has some very basic closure properties then $\mathcal{M} \vDash$ "there is a well-ordering of reals in $L(Hom^\infty, \mathbb{R})$". As AD implies the reals cannot be well-ordered, $\mathcal{M}$ cannot satisfy Sealing.

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Our improved upper bound for Sealing puts its consistency strength well within the short extender region. Though Sealing will not be a consequence of any such large cardinal hypothesis in this region.

Inner Model Problem (IMPr) and the Core Model Induction

Our interpretation of IMPr is influenced by John Steel's view on Gödel's Program (see [Ste14]). In a nutshell, the idea is to develop a theory that connects various foundational frameworks such as Forcing Axioms, Large Cardinals, Determinacy Axioms etc with one another. In this view, IMPr is the bridge between all of these natural frameworks and IMPr needs to be solved under variety of hypotheses, such as PFA, the failure of Jensen's $\square$ principles, the existence of strong ideals etc. Our primary tool for solving IMPr in large-cardinal-free contexts is the Core Model Induction (CMI).

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In the earlier days, CMI was perceived as an inductive method for proving determinacy in models such as $L(\mathbb{R})$. The goal was to prove that $L_\alpha(\mathbb{R}) \vDash \mathsf{AD}$ by induction on α. In those earlier days, which is approximately the period 1995-2010, the method worked by establishing intricate connections between large cardinals, universally Baire sets and determinacy.

Derived Model Theorem and the Core Model Induction

Recall Woodin's Derived Model Theorem. A typical situation works as follows. Suppose λ is a limit of Woodin cardinals and $g \subseteq Coll(\omega, <\lambda)$ is generic. Let $\mathbb{R}^* = \cup_{\alpha<\lambda}\mathbb{R}^{V[g\cap Coll(\omega,\alpha)]}$. Working in $V(\mathbb{R}^*)$, let $Hom = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \vDash \mathsf{AD}\}$. Then

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In Woodin's theorem, *Hom* is *maximal* as there are no more (strongly) determined sets in the universe that are not in *Hom*. If one assumes that λ is a limit of strong cardinals then *Hom* above is just $Hom_\infty^{V(\mathbb{R}^*)}$.

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The aim of CMI is to do the same for other natural set theoretic frameworks, such as forcing axioms, combinatorial statements etc. Suppose T is a natural set theoretic framework and $V \vDash T$. Let κ be an uncountable cardinal. One way to perceive CMI is the following.

(CMI at κ) Saying that one is doing Core Model Induction at κ means that for some $g \subseteq Coll(\omega, \kappa)$, in $V[g]$, one is proving that $L(Hom^\infty, \mathbb{R}) \vDash \mathrm{AD}^+$.

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In both cases, the aim might be less ambitious. It might be that one's goal is to just produce $\Gamma \subseteq Hom^\infty$ such that $L(\Gamma, \mathbb{R})$ is a determinacy model with desired properties.

UB – Covering

Suppose we do our CMI below κ. The current methodology for proving that $\boxed{\mathrm{HOD}^{L(Hom^\infty,\mathbb{R})}}$ has the desired large cardinals is via a failure of certain covering principle involving $\mathrm{HOD}^{L(Hom^\infty,\mathbb{R})}$. Set $\mathcal{H}^- = (\mathrm{HOD}|\Theta)^{L(Hom^\infty,\mathbb{R})}$.

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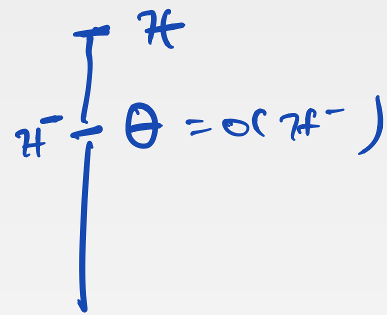
$\Theta = \sup\{\alpha : \exists \pi : \mathbb{R} \twoheadrightarrow \alpha \text{ surjection}, \pi \in L(Hom^\infty, \mathbb{R})\}$

We simply let $\mathcal{H}$ be the union of all *hod mice* extending $\mathcal{H}^-$, projecting to $o(\mathcal{H}^-)$, whose countable submodels have iteration strategies in $L(Hom^\infty, \mathbb{R})$.



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Let now $g \subseteq Coll(\omega,\kappa)$ be V-generic. Because $|V_\kappa| = \kappa$, we have that $|\mathcal{H}^-|^{V[g]} = \aleph_0$ and $|\mathcal{H}|^{V[g]} \leq \aleph_1$. Letting $\eta = Ord \cap \mathcal{H}$,

$$L(Hom^\infty_g, \mathbb{R}_g) \vDash \text{“there is an } \eta\text{-sequence of distinct reals".}$$

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$$L(Hom_g^\infty, \mathbb{R}_g) \models \text{“there is an } \eta\text{-sequence of distinct reals".}$$

Assuming Sealing, we get that $\eta < \omega_1$ as under Sealing, $L(Hom_g^\infty, \mathbb{R}_g) \models \mathrm{AD}$, and under AD there is no ω_1-sequence of reals. Therefore, in V, $\eta < \kappa^+$ as we have that $(\kappa^+)^V = \omega_1^{V[g]}$. Letting now

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Sealing **implies that** UB – Covering **fails** at measurable cardinals. A similar argument can be carried out by only assuming that κ is a singular strong limit cardinal. when κ is singular or measurable.

Sealing and UB – covering

The argument that has been used to show that $\mathcal{H}$ has large cardinals proceeds as follows. Pick a target large cardinal ϕ. Assume $\mathcal{H} \vDash \forall\gamma\neg\phi(\gamma)$. Thus far, in many major applications of the CMI, the facts that

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One path forward

One way to move forward with CMI past Sealing is to develop techniques for building **third order canonical objects, objects that are canonical subsets of** Hom^{∞}. CMI should be viewed as a technique for proving that certain type of covering holds rather than a technique for showing that HOD has large cardinals.

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Conjecture

PFA *implies there is some* $\Gamma \subset \mathcal{P}(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+$ *and let* $\mathcal{H} = HOD^{L(\Gamma,\mathbb{R})}$, *the* $\mathcal{H} \vDash$ *"there is a superstrong cardinal".*

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Assume NLE *and suppose there are unboundedly many Woodin cardinals and strong cardinals. Let* κ *be a limit of Woodin cardinals and strong cardinals such that either* $cof(\kappa) = \kappa$ *or* $cof(\kappa) = \omega$. *Then there is a transitive model* M *of* ZFC − Powerset *such that*

1. $cof(Ord \cap M) \geq \kappa$,
2. M *has a largest cardinal* ν,
3. *for any* $g \subseteq Coll(\omega, <\kappa)$, *letting* $\mathbb{R}^* = \bigcup_{\alpha<\kappa} \mathbb{R}^{V[g \cap Coll(\omega,\alpha)]}$, *in* $V(\mathbb{R}^*)$,

$$L(M, \bigcup_{\alpha<\nu} (M|\alpha)^\omega, Hom^\infty, \mathbb{R}) \vDash \mathsf{AD}.$$

One path forward

One way to move forward with CMI past Sealing is to develop techniques for building **third order canonical objects, objects that are canonical subsets of** Hom^∞. CMI should be viewed as a technique for proving that certain type of covering holds rather than a technique for showing that HOD has large cardinals.

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Hierarchy relations

Large Cardinals	Determinacy	HOD	Combinatorial Theories
Supercompact			PFA, MM
WLW		WLW	ω_1 is (str/super)compact
?	LSA — over — UB	excellent	Sealing
?	LSA	lsa-type hod pair	
Θ reg-hypo	$\mathsf{AD}_{\mathbb{R}}+\Theta$ regular	Regular lim of Wdns	ω_1 is $\mathcal{P}(\mathbb{R})$-spct, MM(c)
non-domestic	$\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$	ω_1 Woodins	ω_1 is $\mathcal{P}(\mathbb{R})$-str.cpct.
$\mathsf{AD}_{\mathbb{R}}$-hypo	$\mathsf{AD}_{\mathbb{R}}$	ω Woodins	
ω Woodins	AD	1 Woodin	$\mathsf{AD}+\omega_1$ is $\mathbb{R}$-str.cpct.

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The Largest Suslin Axiom, abbreviated as LSA, is the conjunction of the following statements:

1. AD^+.
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The result above improves significantly the previous consistency theorem for Sealing by Woodin.

Theorem (Woodin, [Lar04])

Suppose there is a proper class of Woodin cardinals. Let δ be a supercompact cardinal and G be V-generic such that in $V[G]$, $V_{\delta+1}$ is countable. Then Sealing *holds in $V[G]$.*

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Motivated by the following result:

Theorem (Steel, Woodin)

Assume there is a proper class of measurables. The following are equivalent:

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We address the question of whether Sealing and LSA-over-UB are equivalent. The short answer is NO.

Inequivalence (cont.)

We let ile(P) be the set of *inaccessible-length extenders* of P. More precisely ile(P) consists of extenders $E \in P$ such that $P \vDash$ "$lh(E)$ is inaccessible and $V_{lh(E)} = V_{lh(E)}^{Ult(V,E)}$." We say that $\mathcal{P}$ is a *pre-iterable structure* if $\mathcal{P} = (P, \mathrm{ile}(P))$ where P is a transitive model of ZFC.

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We say that self-iterability holds if the following holds in V.

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Theorem (Sargsyan-T., 2018-2019, [ST19c])

Assume self-iterability holds, and suppose there is a class of Woodin cardinals and a strong cardinal. Let κ be the least strong cardinal of V and let $g \subseteq Coll(\omega, \kappa^+)$ be V-generic. Then $V[g] \vDash$ Sealing.

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The paper [ST19a] is the first step towards this goal; in [ST19a], we have constructed from PFA hod mice (Z-hod pairs) that are stronger than an excellent hybrid mouse.

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Assuming PFA and there is a Woodin cardinal, then there is a canonical model of WLW. The proof is not via CMI methods, but just an observation that the full-backgrounded construction as done in [Nee02] reaches a model of WLW. The Woodin cardinal assumption is important here. The argument would not work if one assumes just PFA and/or a large cardinal milder than a Woodin cardinal, e.g. a measurable cardinal or a strong cardinal.

The paper [ST19a] is the first step towards this goal; in [ST19a], we have constructed from PFA hod mice (Z-hod pairs) that are stronger than an excellent hybrid mouse.

Equivalence/equiconsistency at the level of Sealing

Conjecture

Are the following theories equiconsistent?

1. Sealing $+$ *"There is a proper class of Woodin cardinals".*
2. LSA-over-uB $+$ *"There is a proper class of Woodin cardinals".*
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Conjecture

Suppose there are unboundedly many Woodin cardinals and the class of measurable cardinals is stationary. Then the following are equivalent.

1. Sealing.
2. Sealing^{+}.
3. Weak Sealing.
4. Sealing^{-}.
5. Tower Sealing.

Equiconsistency

Definition

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For the ($\Leftarrow$) direction: show Sealing and LSA – over – UB holds in $\mathcal{P}^{Coll(\omega,\mathcal{P}_0)}$.

UBH and guB strategies

The Unique Branch Hypothesis (UBH) is the statement that every non-dropping plus-2 iteration tree $\mathcal{T}$ on V has at most one cofinal well-founded branch. The Generic Unique Branch Hypothesis (gUBH) says that UBH holds in all set generic extensions.

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We say $(\mathcal{P}, \Psi)$ is an *iterable pair* if $\mathcal{P}$ is a pre-iterable structure and Ψ is a strategy for it. Given a strong limit cardinal κ and $F \subseteq Ord$, set

$$W_\kappa^{\Psi,F} = (H_\kappa, F \cap \kappa, \mathcal{P}|\kappa, \Psi_{\mathcal{P}|\kappa} \restriction H_\kappa, \in).$$

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Given a structure Q in a language extending the language of set theory with a transitive universe, and an $X \prec Q$, we let M_X be the transitive collapse of X and $\pi_X : M_X \to Q$ be the inverse of the transitive collapse. In general, the preimages of objects in X will be denoted by using X as a subscript. Suppose in addition $Q = (R, ...\mathcal{P}...)$ where $\mathcal{P}$ is a pre-iterable structure and Φ is an iteration strategy of $\mathcal{P}$. **We will then write $X \prec (Q|\Phi)$ to mean that $X \prec Q$ and the strategy of $\mathcal{P}_X$ that we are interested in is Φ^{π_X}, the pullback of Φ via π_X. We set $\Lambda_X = \Phi^{\pi_X}$.**

UBH and guB strategies (cont.)

Definition

We say Ψ is a *generically universally Baire (guB) strategy* for a pre-iterable $\mathcal{P} = (P, \vec{E})$ if there is a formula $\phi(x)$ in the language of set theory augmented by three relation symbols and $F \subseteq Ord$ such that for every inaccessible cardinal κ and for every countable

$$X \prec (W_\kappa^{\Psi,F} | \Psi_{\mathcal{P}|\kappa})$$

whenever

- (a) $g \in V$ is M_X-generic for a poset of size $< \kappa_X$ and
- (b) $\mathcal{T} \in M_X[g]$ is such that for some inaccessible $\eta < \kappa_X$, $\mathcal{T}$ is an iteration of $\mathcal{P}_X|\eta$,

the following conditions hold:

1. if $lh(\mathcal{T})$ is a limit ordinal and $\mathcal{T} \in dom(\Lambda_X)$ then $\Lambda_X(\mathcal{T}) \in M_X[g]$,
2. $\mathcal{T}$ is according to Λ_X if and only if $\mathcal{M}_X[g] \vDash \phi[\mathcal{T}]$.

We say that (ϕ, F) is a generic prescription of Ψ.

Proof outline of Sealing

Theorem (Sargsyan-T., 2018-2019, [ST19c])

Assume self-iterability holds, and suppose there is a class of Woodin cardinals and a strong cardinal. Let κ be the least strong cardinal of V and let $g \subseteq Coll(\omega, \kappa^+)$ be V-generic. Then $V[g] \vDash$ Sealing.

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Working in $V[g * h]$, let $D(h, \eta, \delta, \lambda)$ be the club of countable

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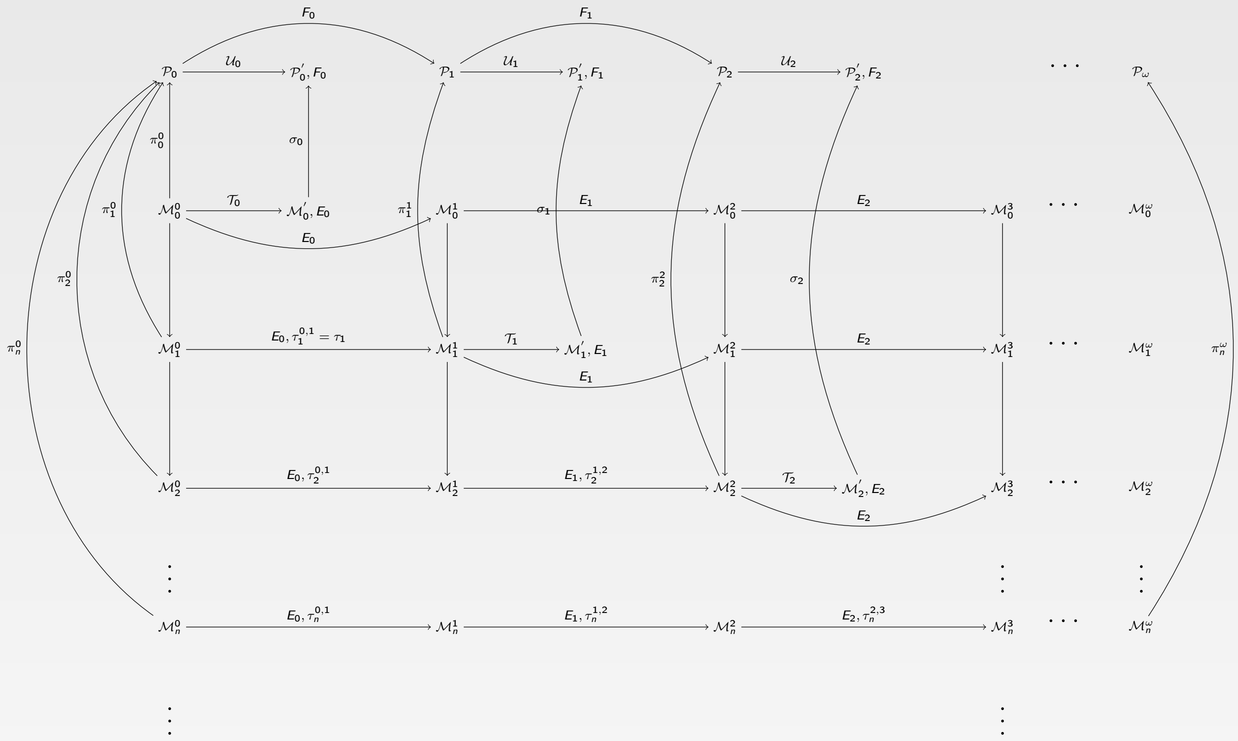

Figure: Diagram of the main argument

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To get Sealing, dovetail two such iterations.

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Thank you!