Classification of P-oligomorphic permutation groups

Conjectures of Cameron and Macpherson

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Université Gustave Eiffel (Marne-la-Vallée)

Models and Sets seminar, 28th April 2021

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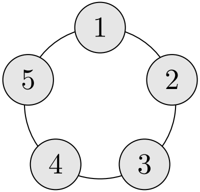
Cyclic permutation group $G = \mathcal{C}_5 =< (1\ 2\ 3\ 4\ 5) >$

$\rightarrow$ natural action on the five-pearl necklace



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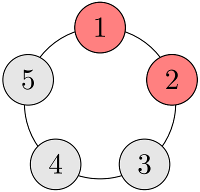

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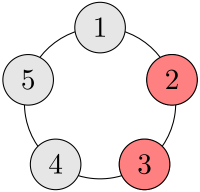

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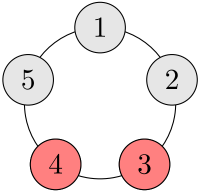

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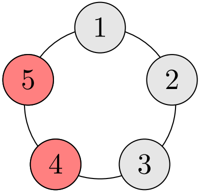

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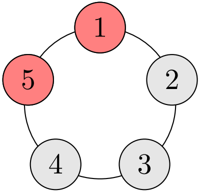

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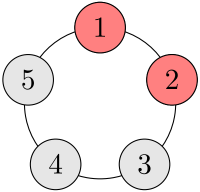

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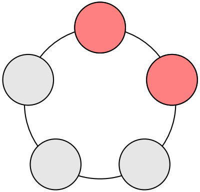

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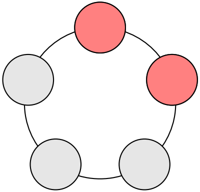

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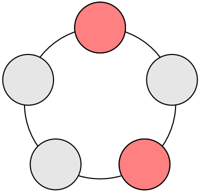

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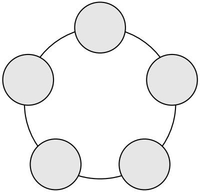

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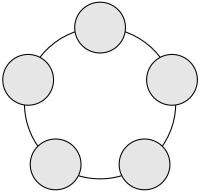

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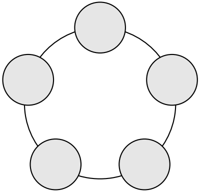

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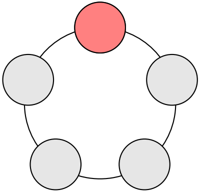

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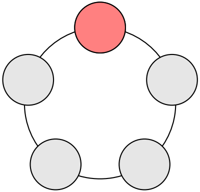

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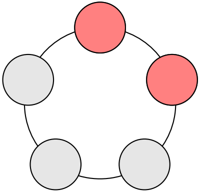

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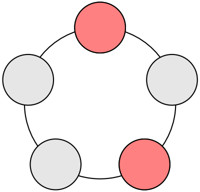

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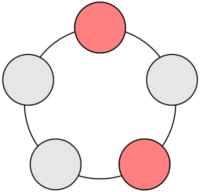

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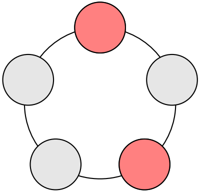

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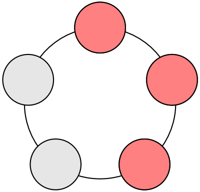

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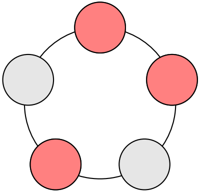

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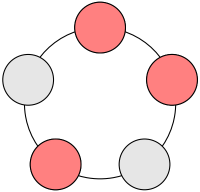

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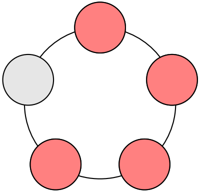

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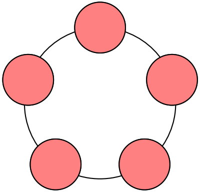

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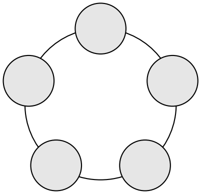

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Example

$$\mathcal{H}_{\mathfrak{S}_\infty}(z) \;=\; 1 + z + z^2 + \cdots \;=\; \frac{1}{1-z}$$

Oligomorphic groups in model theory

Hypothesis

G is **oligomorphic** : φ_G takes only finite values

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A countable first-order structure is $\aleph_0$-categoric if and only if its automorphism group is oligomorphic.

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Structures with a high degree of symmetry.

Conjecture of Cameron

Stronger hypothesis

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Conjecture 1 - Cameron, 70's

G P-**oligomorphic** $\Rightarrow$ $\varphi_G(n) \sim an^k\,,\ \ k \in \mathbb{N}$

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Orbit algebra (Cameron, 80's)

Structure of graded algebra $\mathcal{A}_G = \bigoplus_n \mathcal{A}_n$ on the orbits

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Hilbert series of the graded algebra

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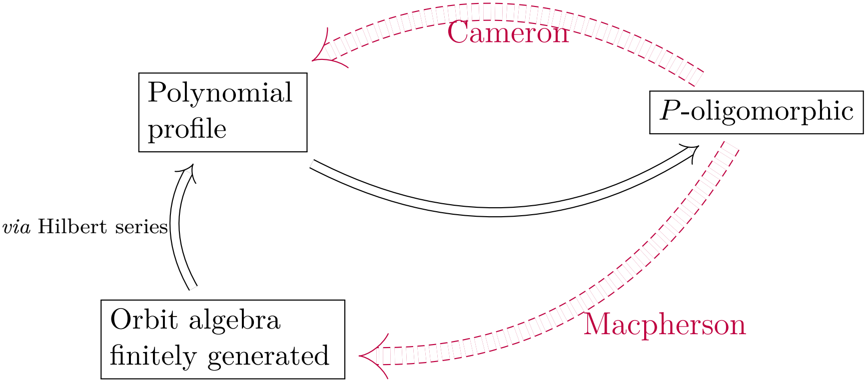
Conjecture 2 (stronger) - Macpherson, 85

G P-oligomorphic $\Rightarrow$ $\mathcal{A}_G$ is finitely generated

Overview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Polynomial profile
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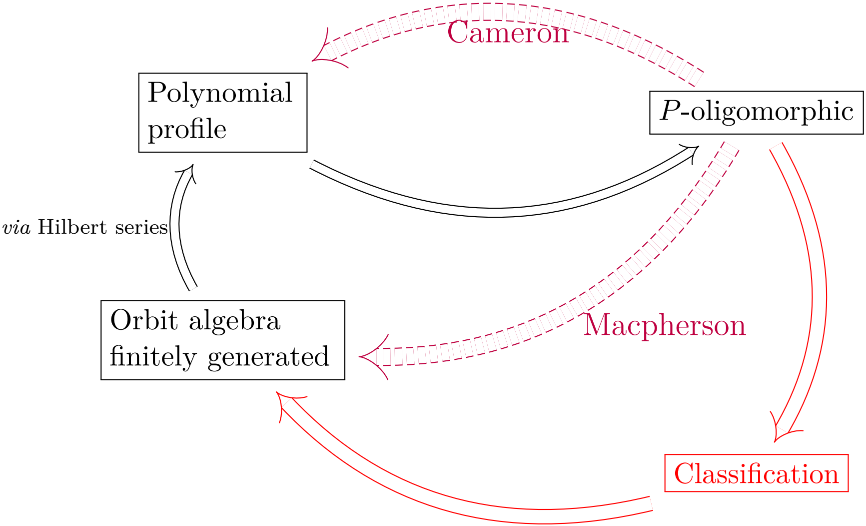

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Conjecture 2 (stronger) - Macpherson, 85

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Theorem (Thiéry, F. 2018)

The orbit algebra of a P-oligomorphic group is finitely generated, and Cohen-Macaulay.

In particular, its profile is polynomial in the strong sense.

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Block system

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- Set partition of the domain into blocks

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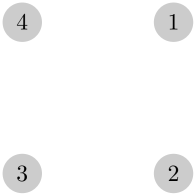
Block systems of C_4



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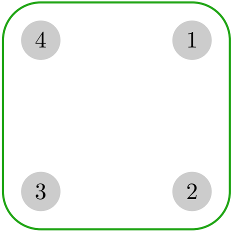

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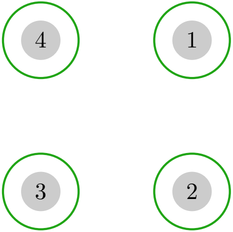

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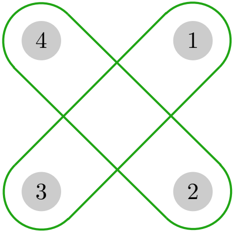

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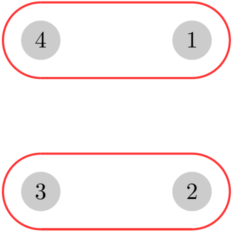
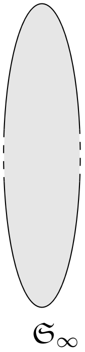
Not a block system $\rightarrow$



The (closed) primitive P-oligomorphic groups



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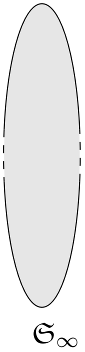
Macpherson:

G P-oligomorphic with no (non trivial) blocks $\Rightarrow \varphi_G(n) = 1 \ \forall n$



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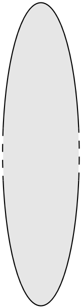

$\mathfrak{S}_\infty$

Theorem (Classification, Cameron)

Only 5 closed groups such that $\varphi_G(n) = 1 \quad \forall\, n$

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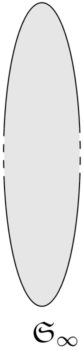

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- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain
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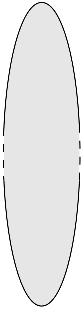

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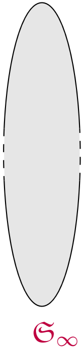
Well known, nice groups (called *highly homogeneous*).
In particular, their orbit algebra is finitely generated.

An infinite example: $\mathfrak{S}_\infty \wr \mathfrak{S}_3$



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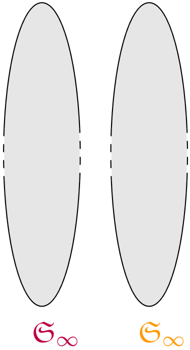

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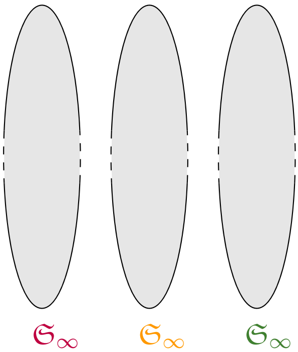

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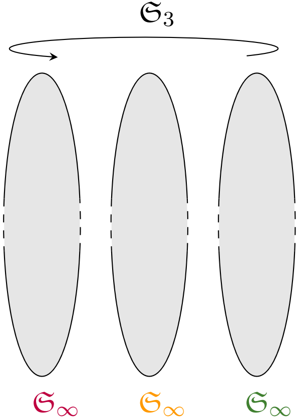

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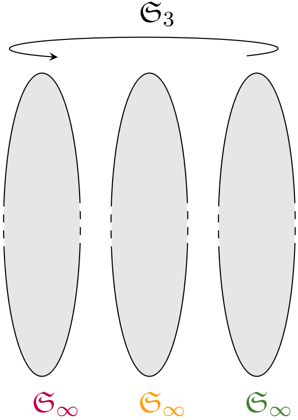

Wreath product

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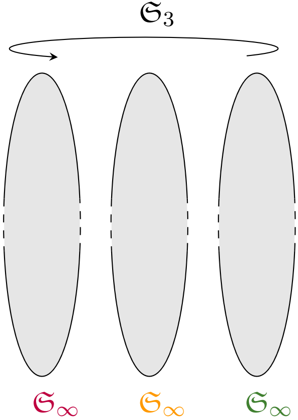

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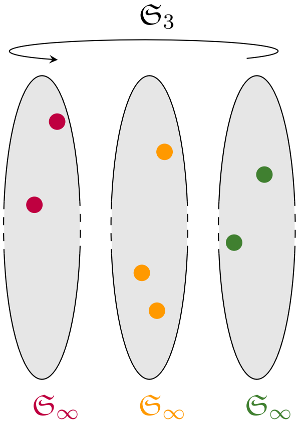

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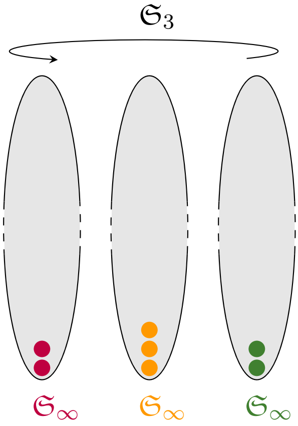

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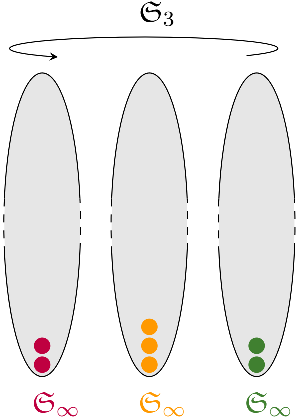

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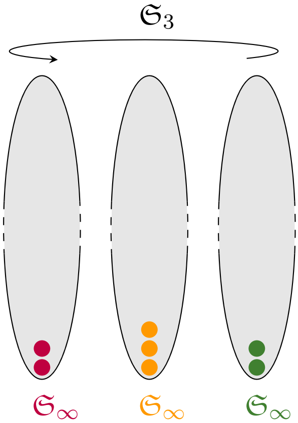

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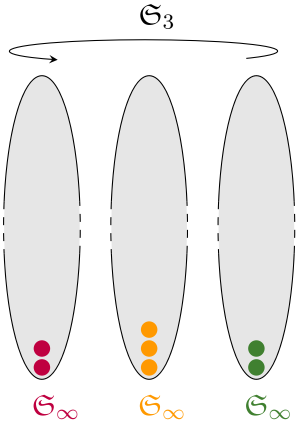

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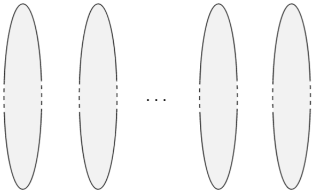
Examples

Integer partitions; combinations; P-partitions...
(with optional length and/or height restrictions)

Further examples

More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathcal{A}_G \simeq \mathbb{Q}[X_1, \ldots, X_m]^H$, the algebra of invariants of H



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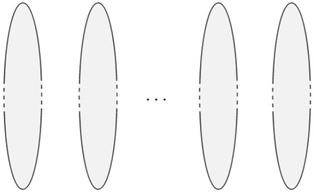

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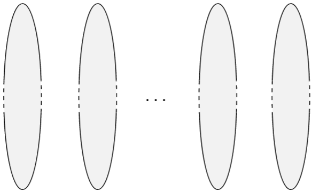

- $G = H \wr \mathfrak{S}_\infty$:
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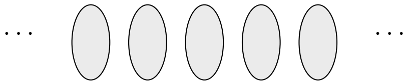

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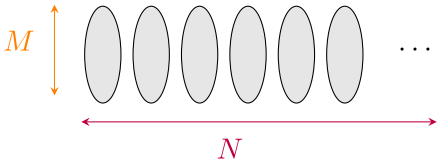

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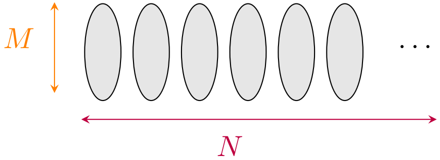

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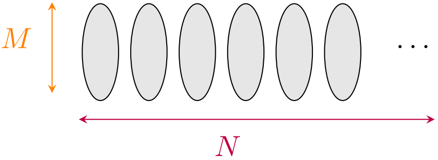

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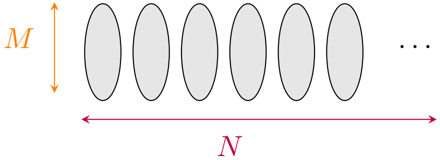

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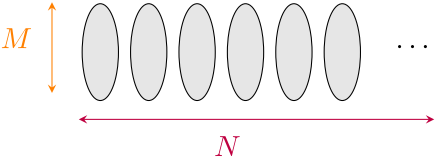

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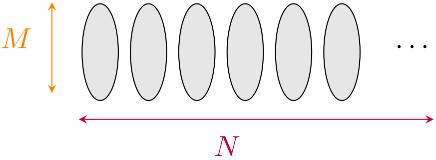

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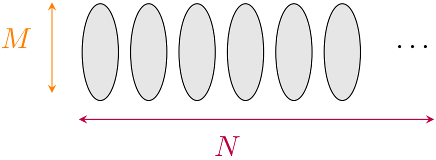

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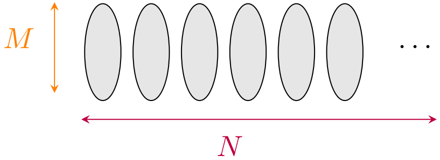

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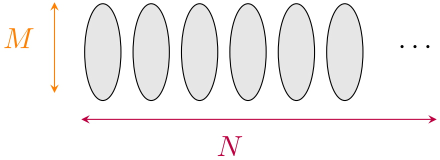

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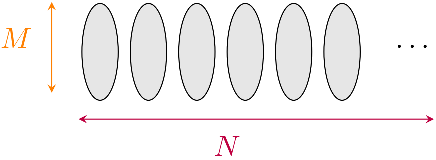

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Better have big finite blocks and/or “small” infinite ones...

Lattices

Lattice

Partially ordered set (poset) with notions of join $\vee$ and meet $\wedge$: any subset has a unique supremum (resp. infimum).

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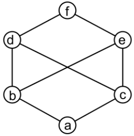

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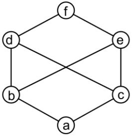
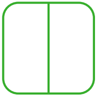


Meet and join in the lattice of set partitions



A

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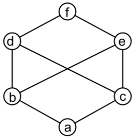

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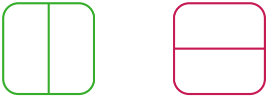

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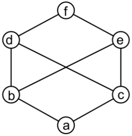

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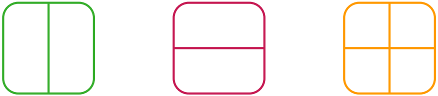

A $\quad$ B $\quad$ $A \wedge B$

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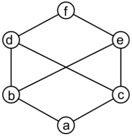

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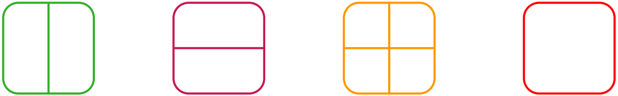

A $\quad B \quad A \wedge B \quad A \vee B$

Lattices of block systems

Lattice of set partitions $\rightarrow$ lattice on block systems

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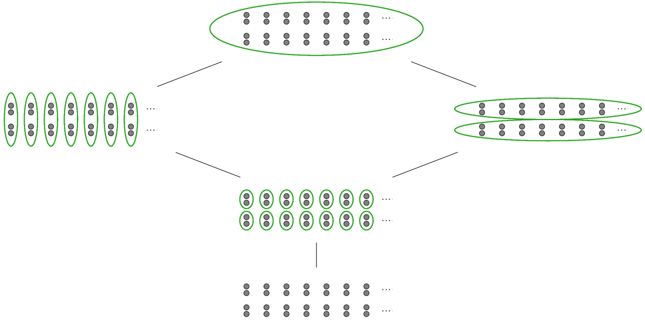
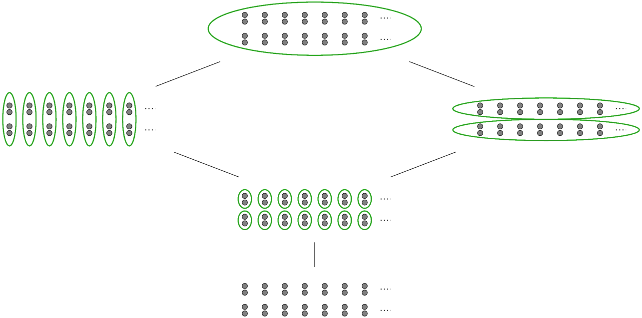


$(\mathcal{C}_2 \times \mathcal{C}_2) \wr \mathfrak{S}_\infty$

Lattices of block systems

Lattice of set partitions $\rightarrow$ lattice on block systems



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Proposition (F.)

- {Systems with $< \infty$ blocks only} = sublattice with maximum
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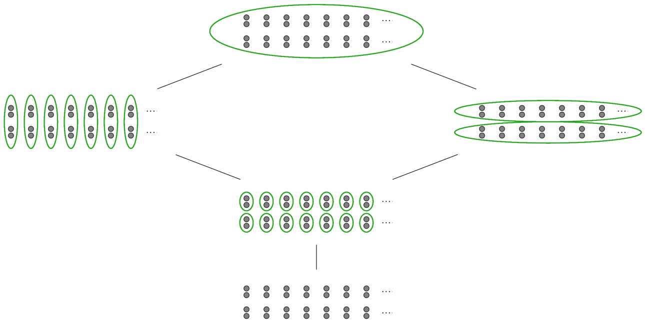
Lattices of block systems

Lattice of set partitions $\to$ lattice on block systems



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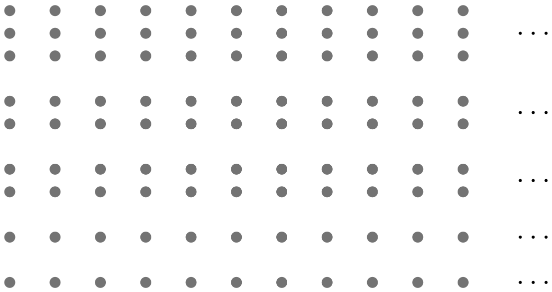
Remark. If G is P-oligomorphic, both of them are actually finite!

The nested block system

Idea



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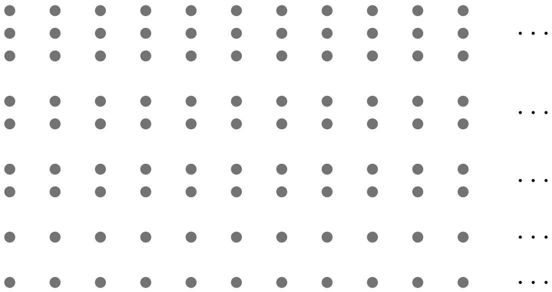
1. Take the *maximal* system of finite blocks



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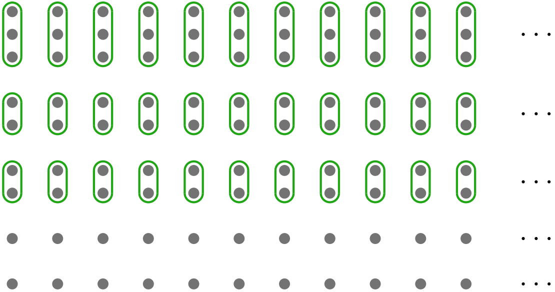

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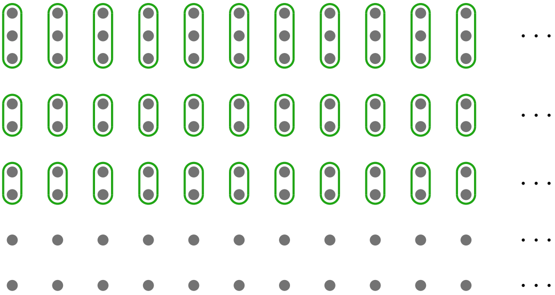

Action on the maximal finite blocks...

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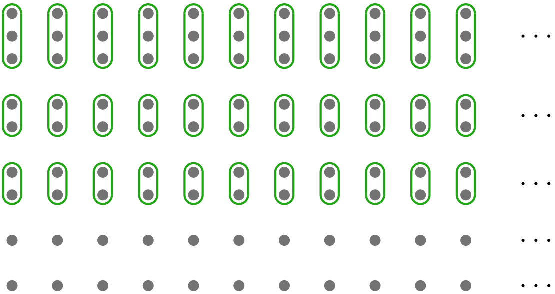

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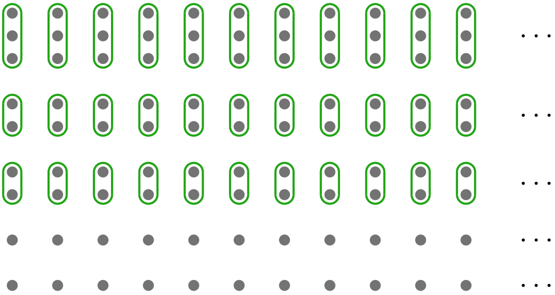

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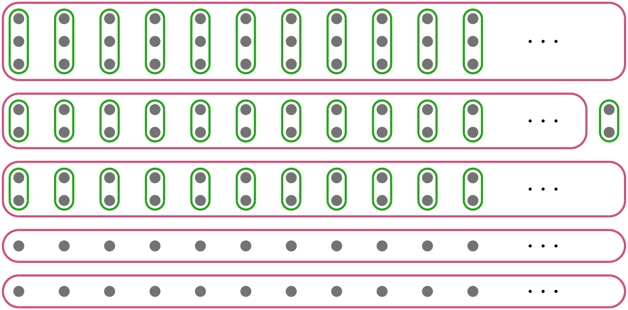

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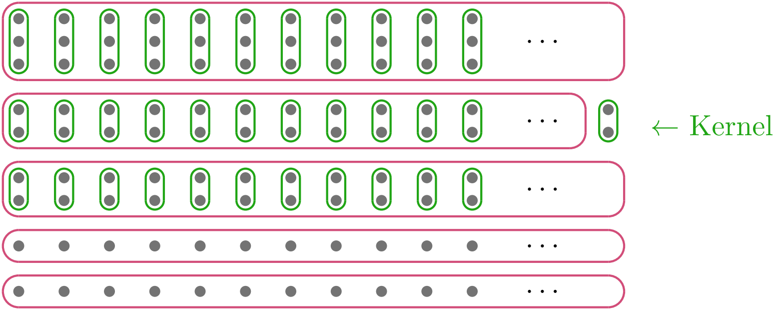

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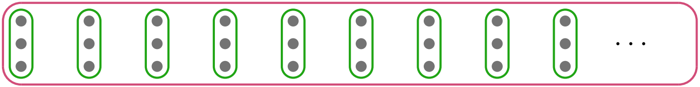

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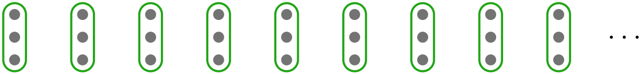

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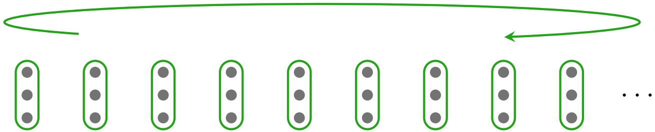

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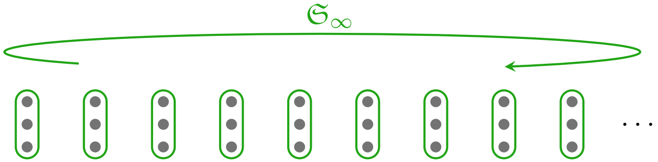

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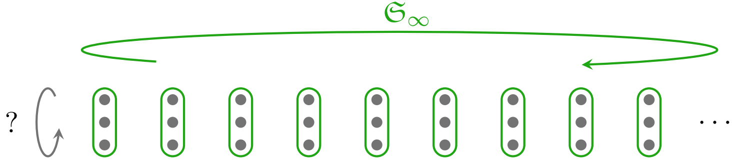

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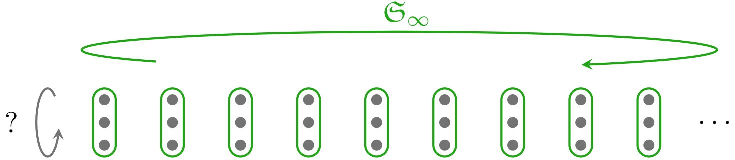

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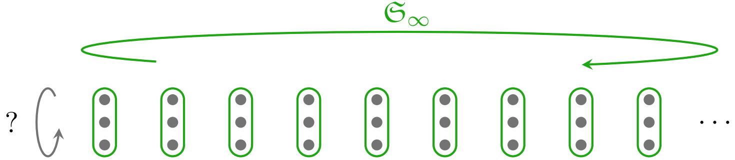

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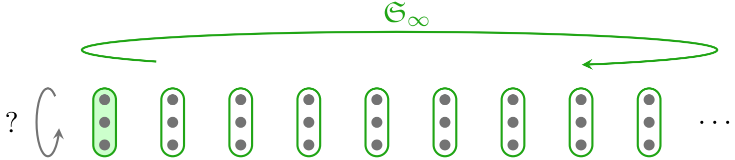

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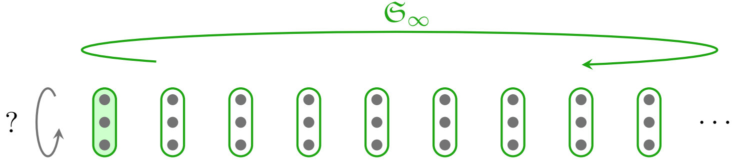

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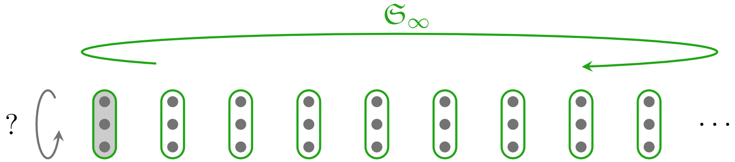

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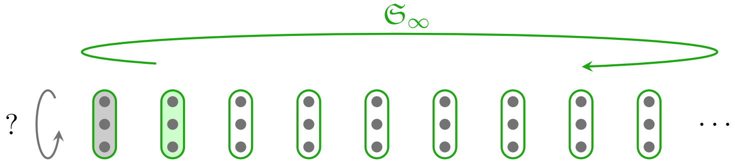

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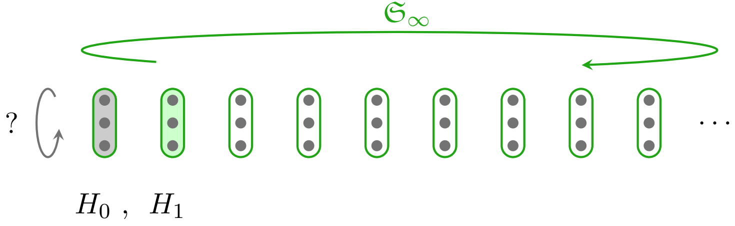

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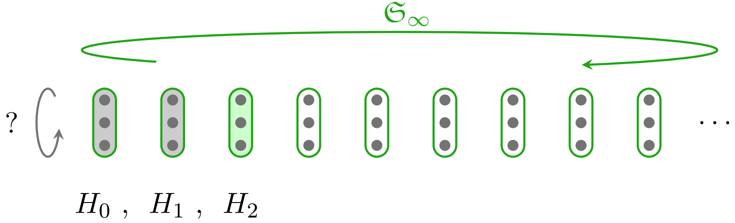

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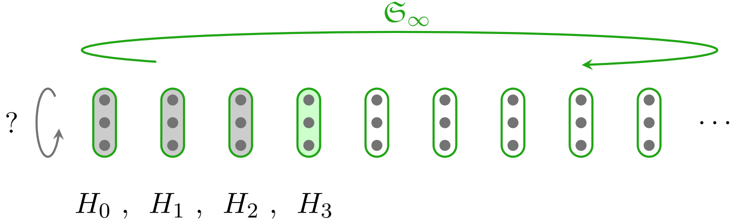

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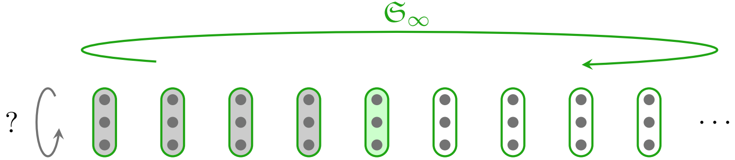

H_0 , H_1 , H_2 , H_3 , H_4

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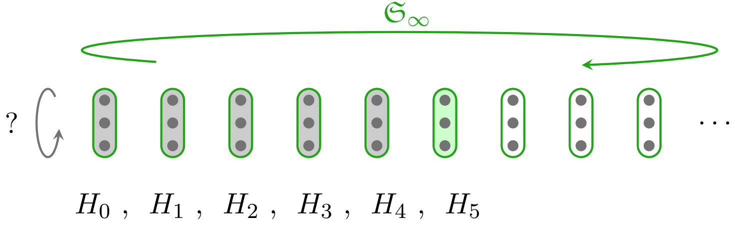

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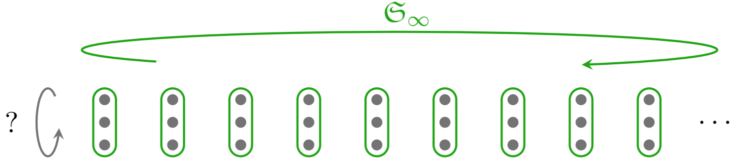

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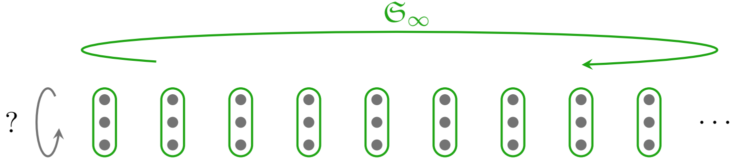

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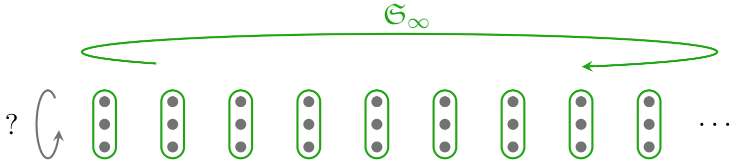

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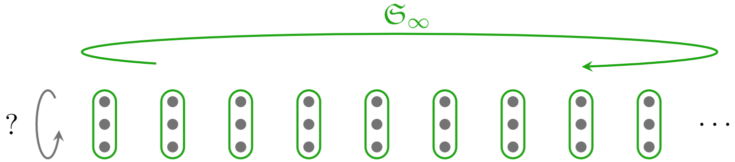

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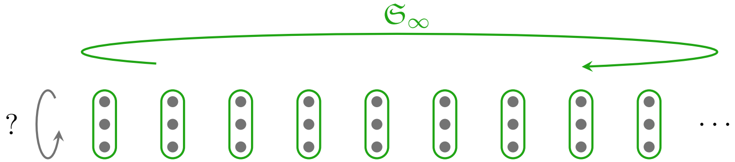

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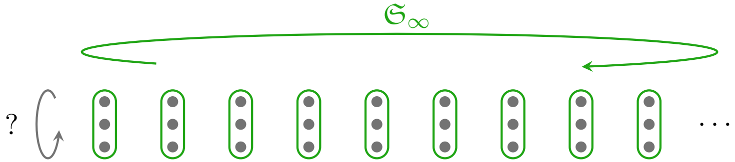

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Notation: $[H_0, H_\infty]$

Subdirect product and synchronization

How to handle synchronizations between blocks ?

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Subdirect product of two groups, or actions

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Remark. The possible synchronizations of a group with another one are linked to its normal subgroups.

The tower determines $\mathrm{Stab}_G(\text{blocks})$

Fact. $\mathrm{Stab}_G(\text{blocks})$ = explicit subdirect product of the H_i

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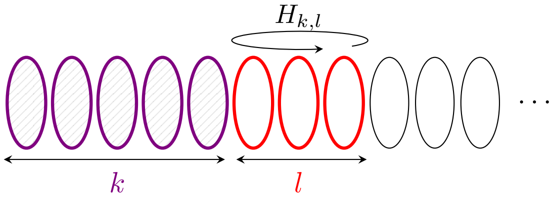

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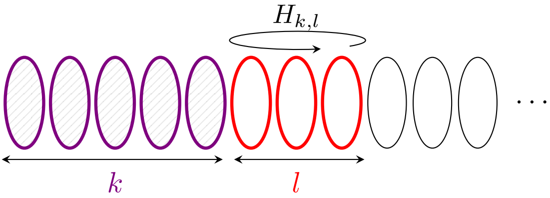

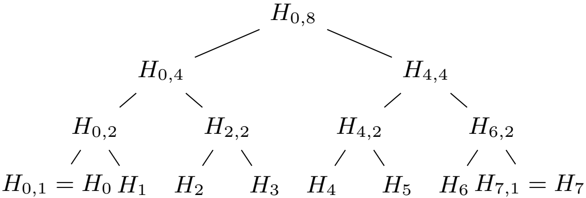

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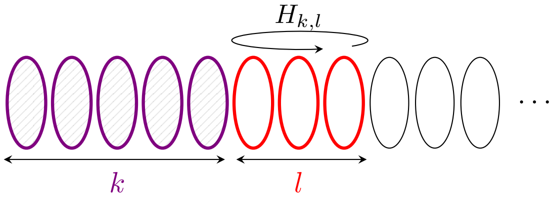

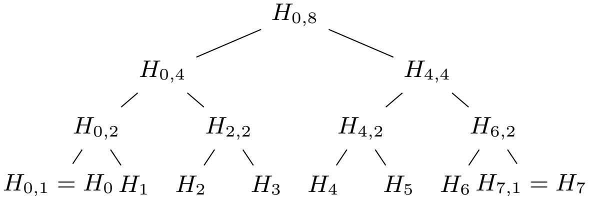

← Recursive subdirect product construction

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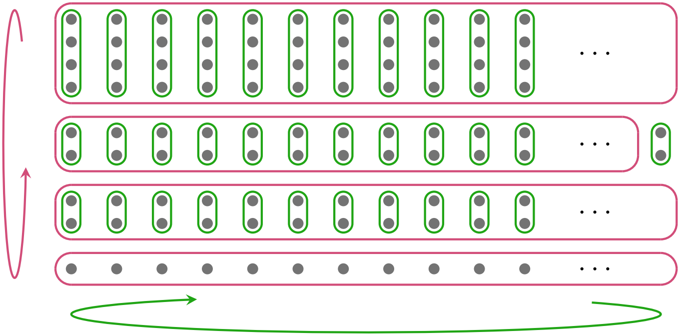
General case: minimal subgroup of finite index



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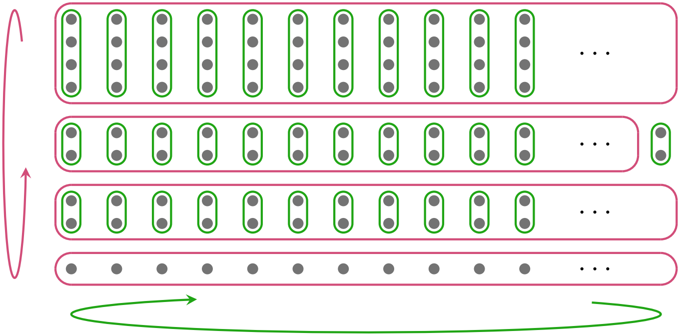

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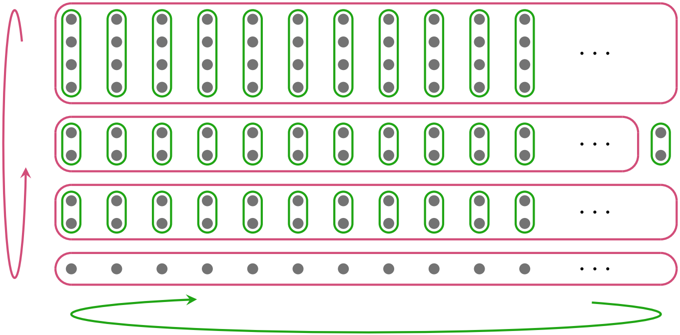

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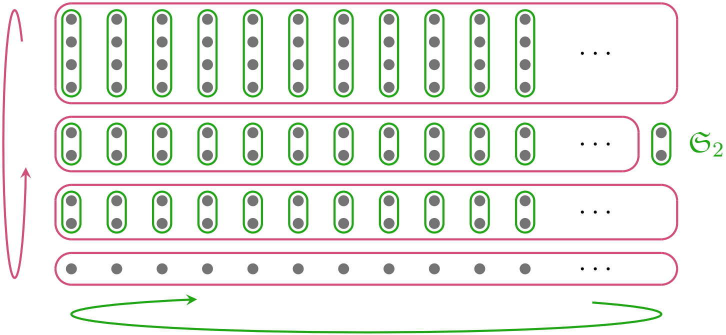

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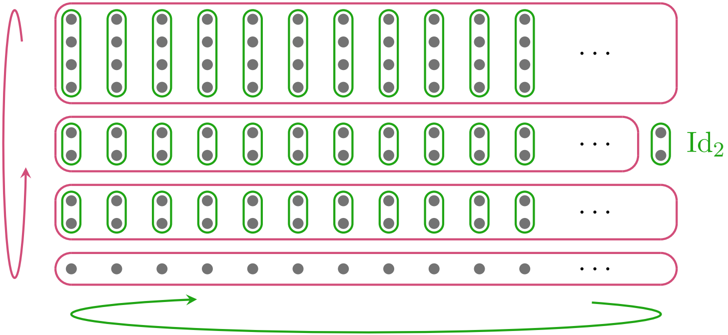

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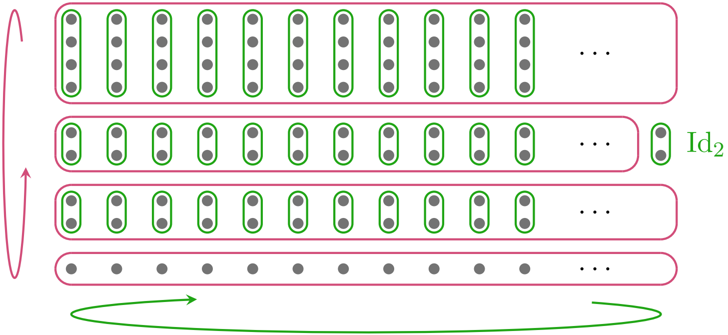

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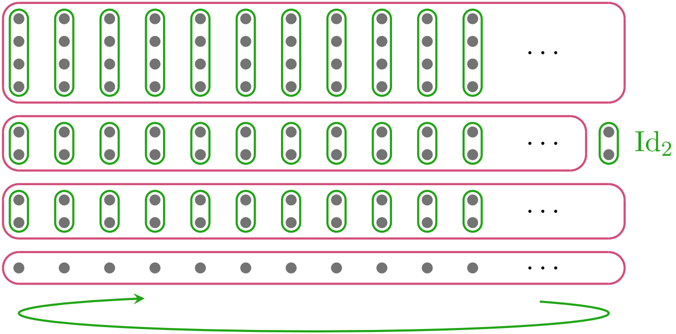

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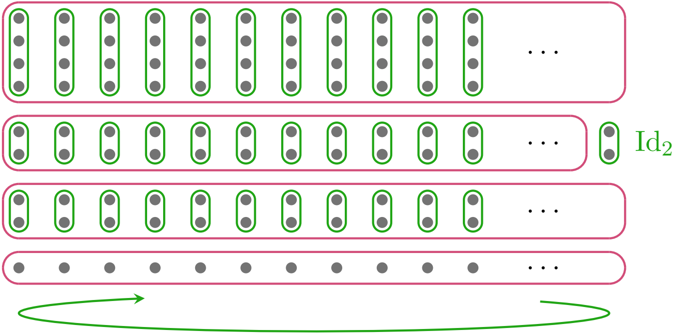

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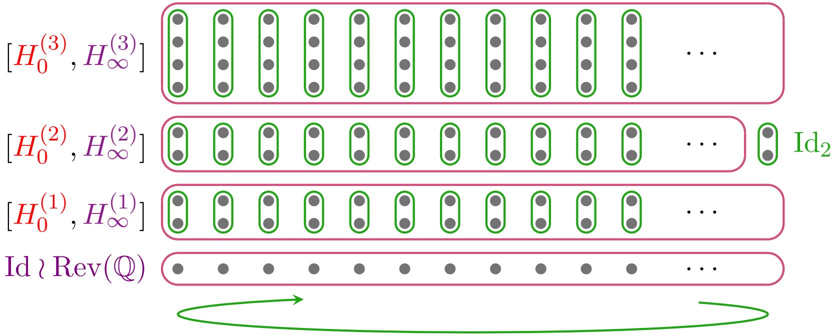

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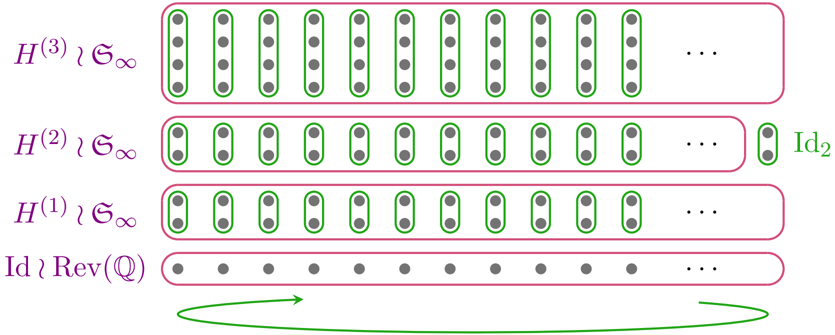

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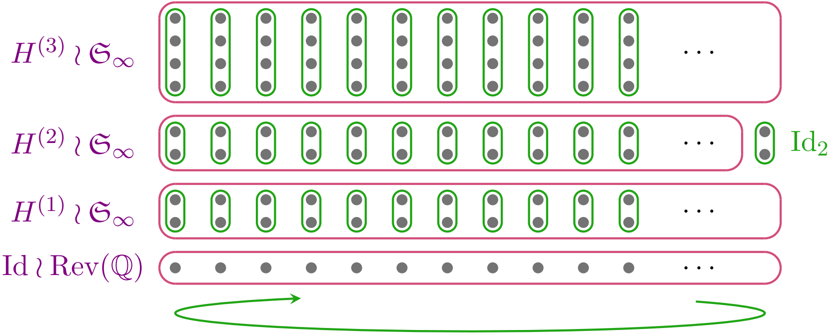

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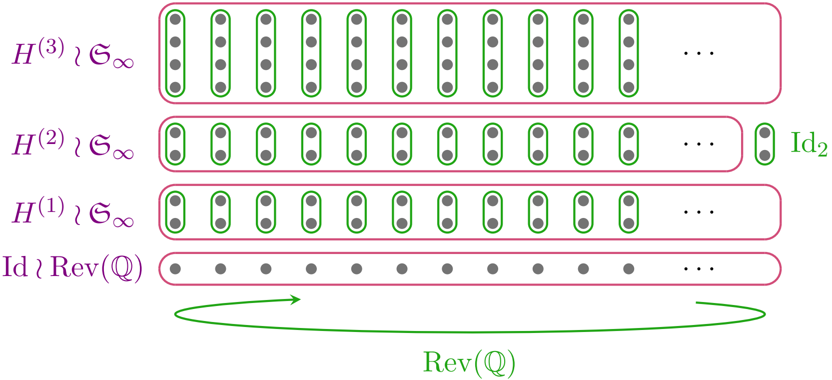

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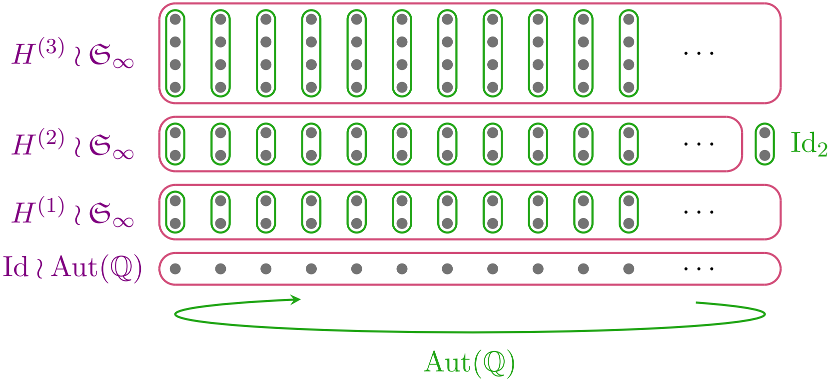

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Which ends the proof of the conjectures !

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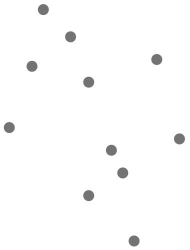

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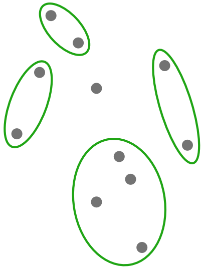

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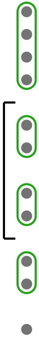

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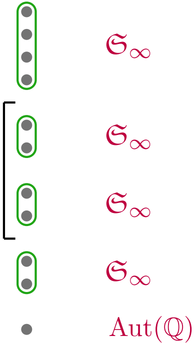

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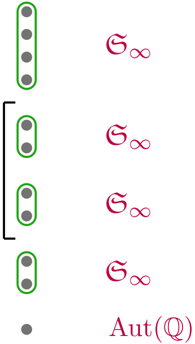

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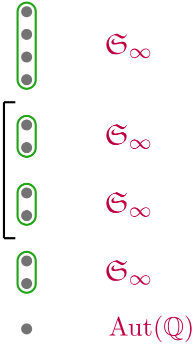

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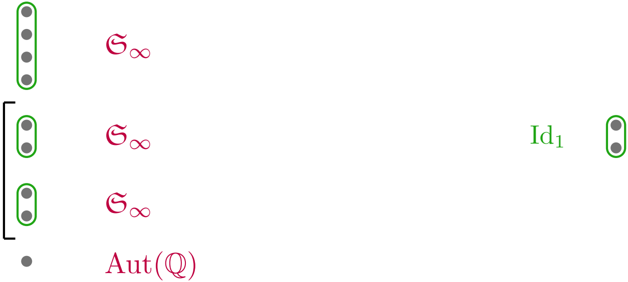

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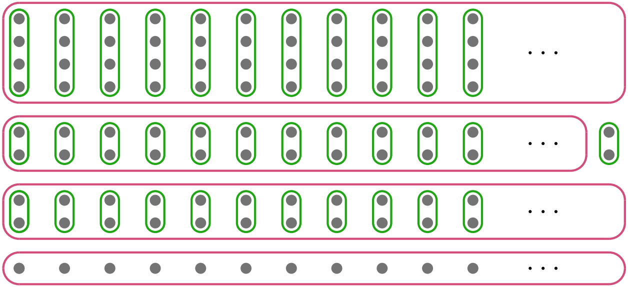

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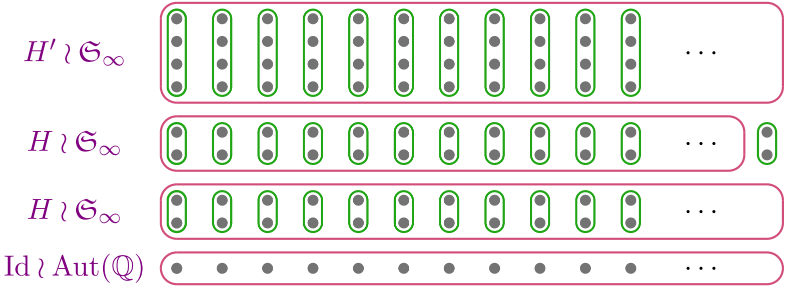

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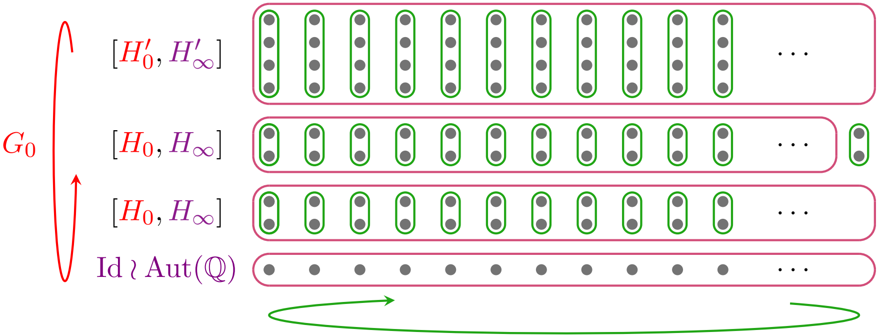

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  - Exponential: wilder primitive groups appear...

Thank you for your attention !

Context

- G permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- Hypothesis: $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron): $\varphi_G(n) \sim an^k$
- Conjecture (Macpherson): finite generation of the orbit algebra

Results

- Both conjectures hold !
- Classification of P-oligomorphic permutation groups
- The orbit algebra is an algebra of invariants (up to some 2-nilpotent elements)

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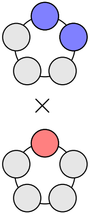

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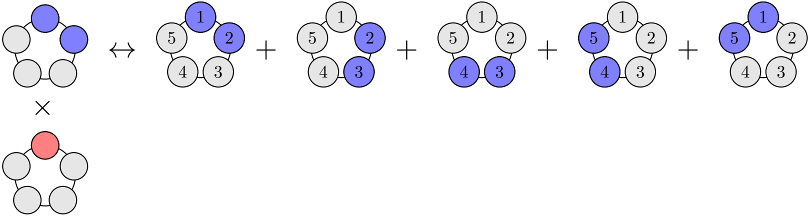

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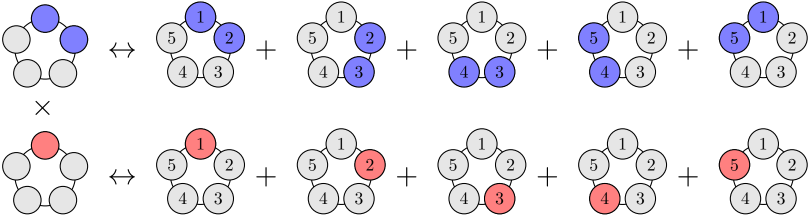

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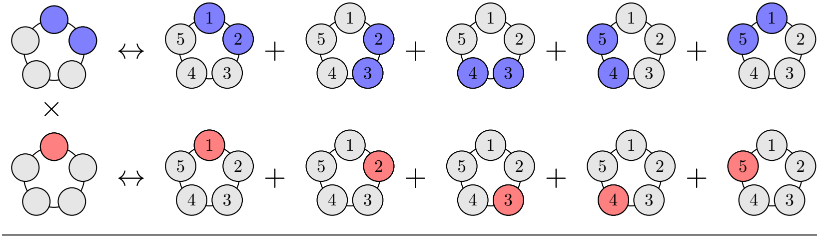

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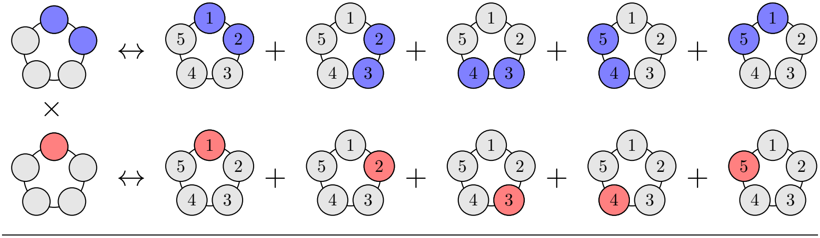

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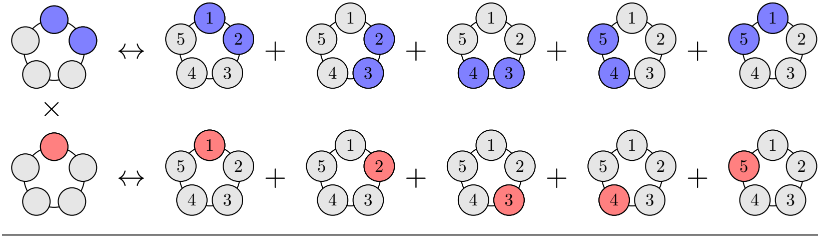

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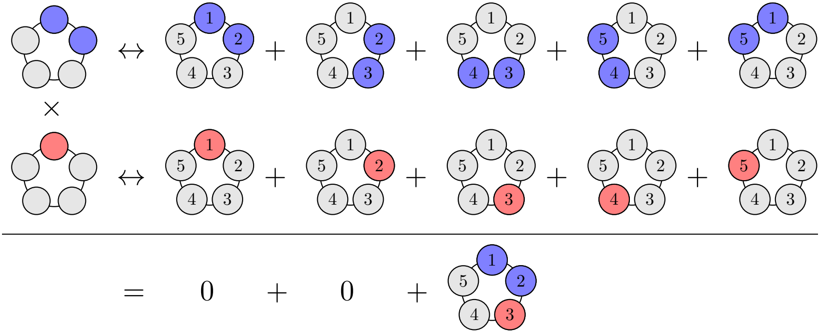

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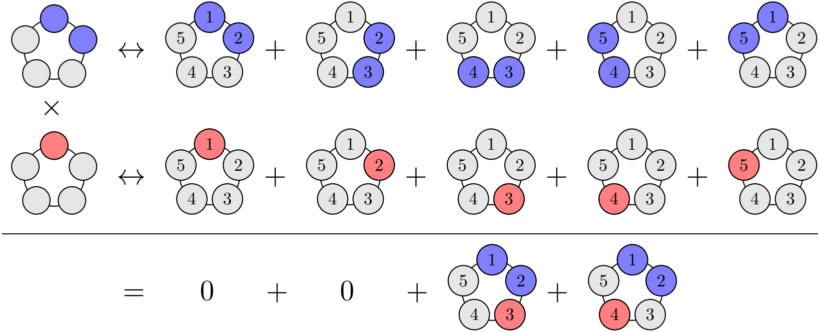

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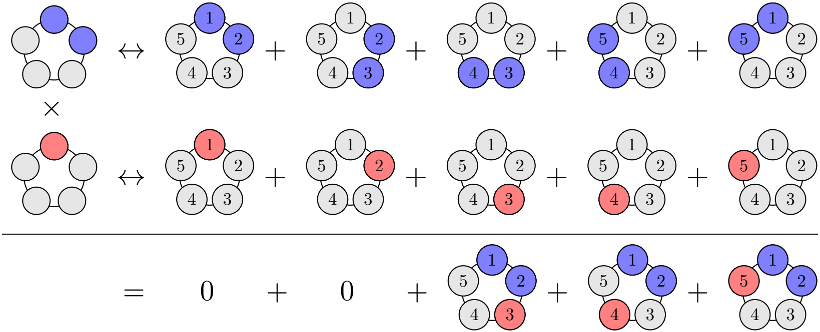

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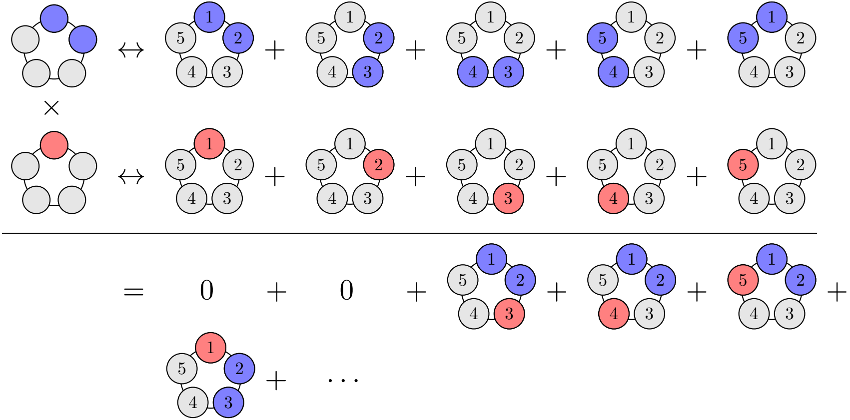

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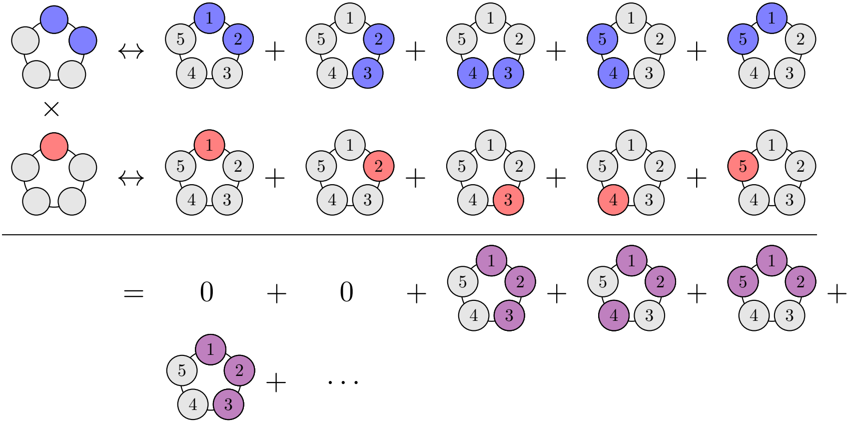

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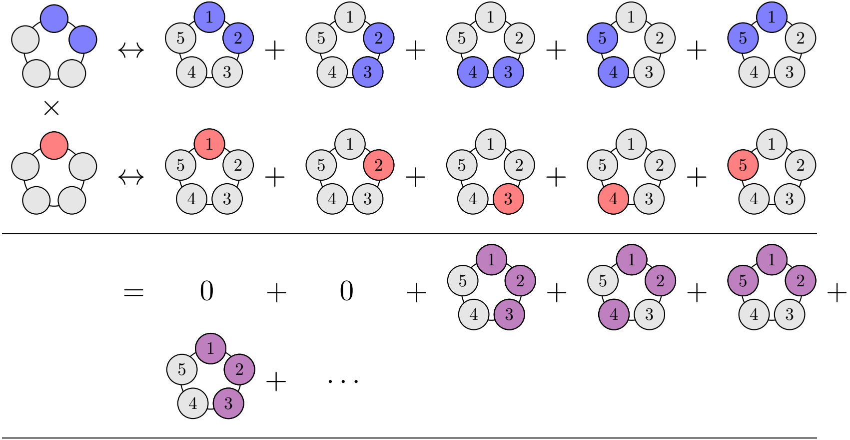

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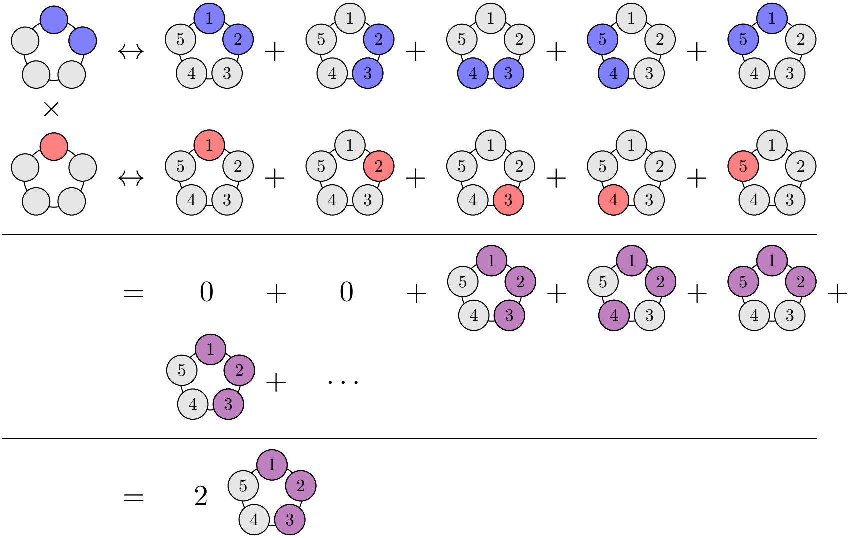

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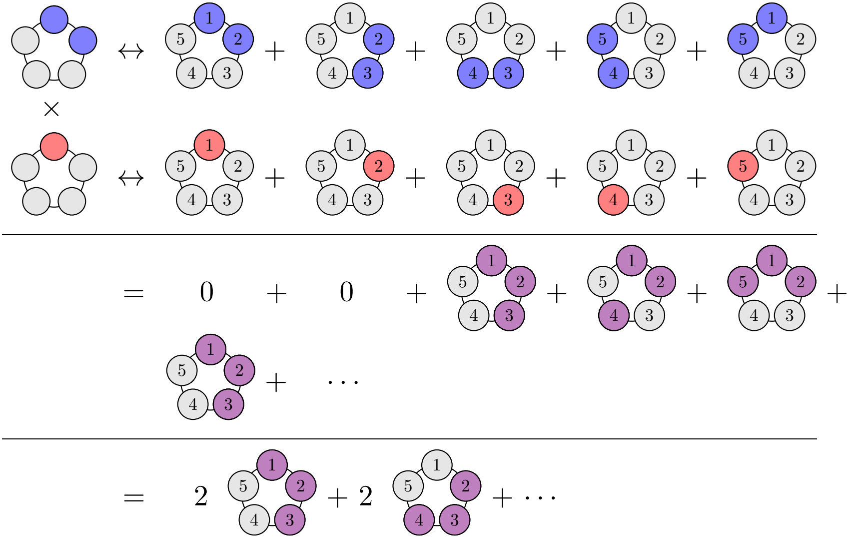

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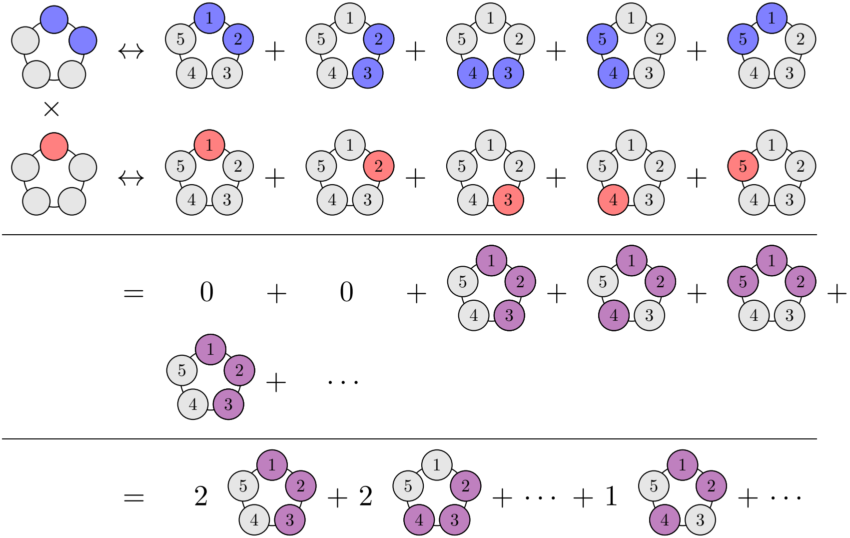

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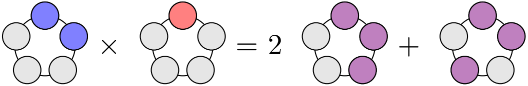

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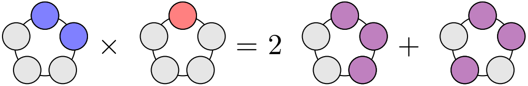

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Product well defined (and graded) on the space of orbits.

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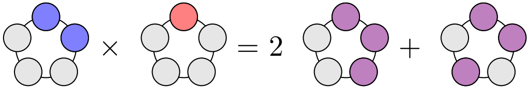

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$\longrightarrow$ **Orbit algebra of a permutation group**

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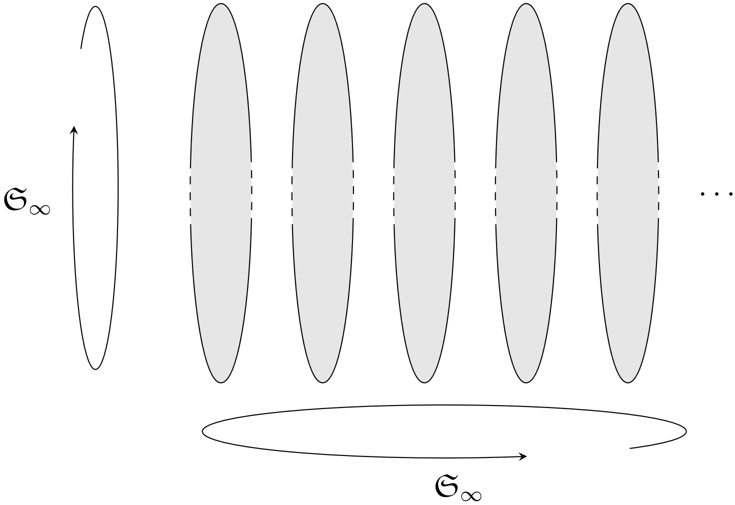

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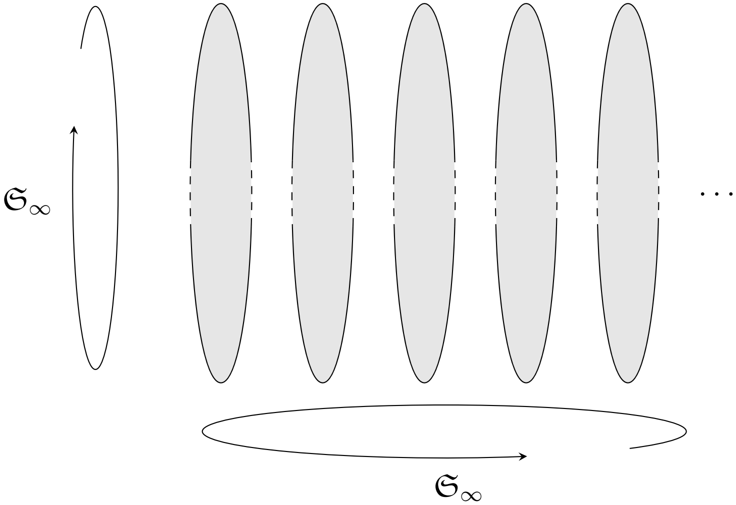

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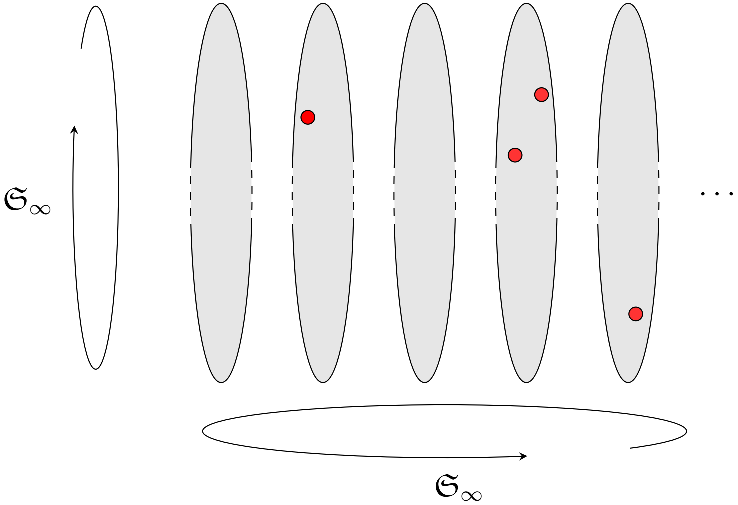

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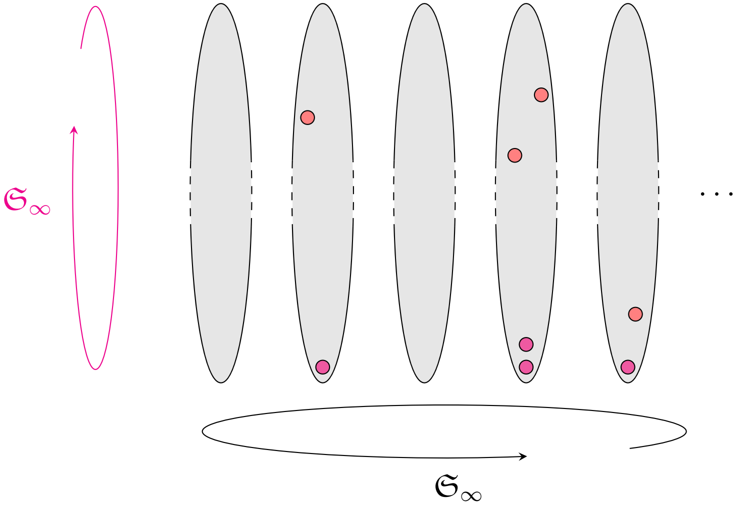

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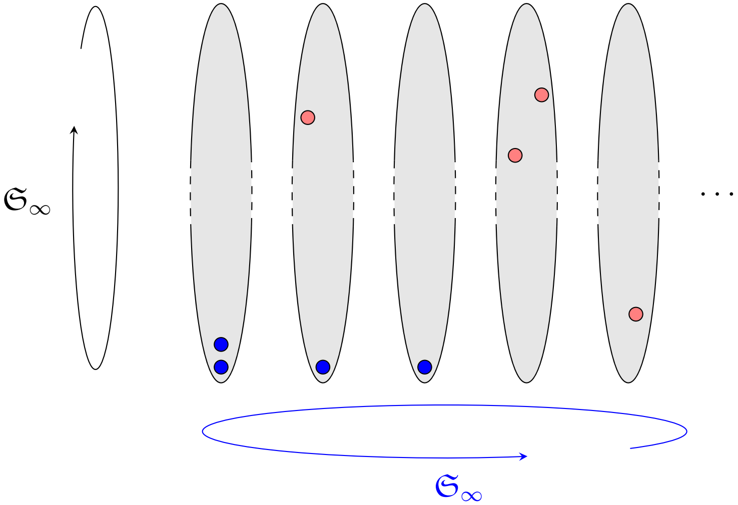

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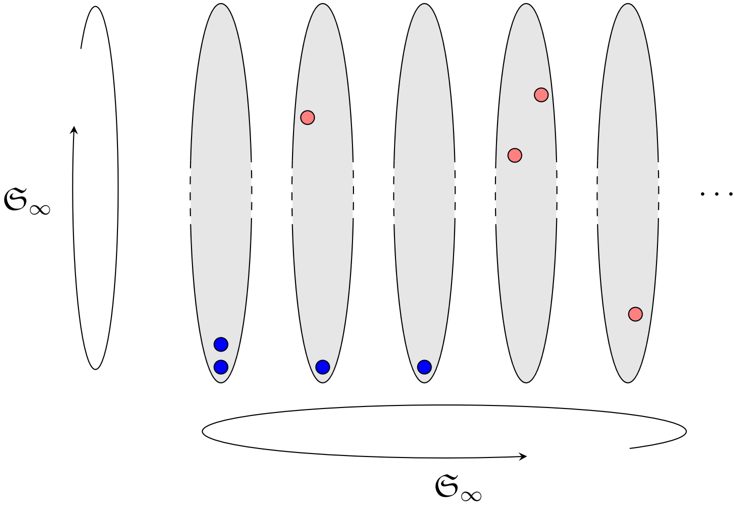
$\varphi_G(n) = p(n)$

An orbit of degree $n \longleftrightarrow$ a partition of n



The tower determines the group (1): "straight $\mathfrak{S}_\infty$"

G contains a set of "straight" swaps of blocks

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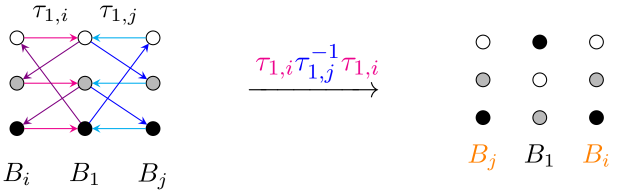

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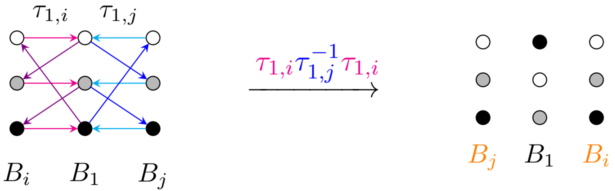
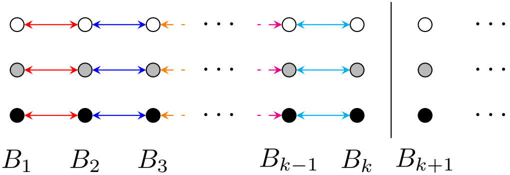




Hence the actions on and within the blocks are independent.

The tower has shape $H_0, \ H, \ H, \ H \ \cdots$

Lemma to prove

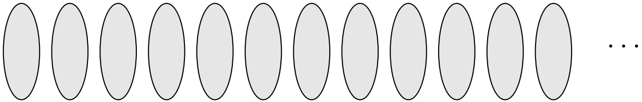
G has tower $H_0 \ H_1 \ H_2 \ H_3 \Rightarrow H_1 = H_2$

Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists \ (1, g, h, k), \ \ h, k \in H_1.$
Let σ be an element of G that permutes "straightforwardly" the first two blocks and fixes the other two.
Conjugation of x by σ in $G \quad \rightarrow \quad y = (g, 1, h, k)$
Then: $x^{-1}y = (g, g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2.

In the infinite case, apply to each restriction to four consecutive blocks of the fixator of the previous ones in G.

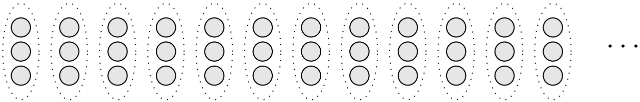
Direct product in the case of finite blocks



Direct product in the case of finite blocks

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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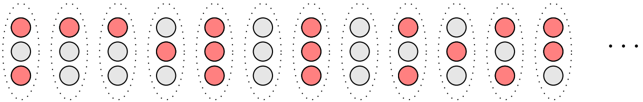

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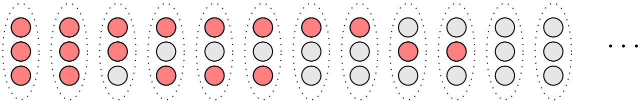

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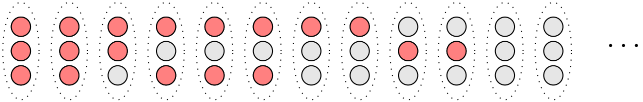

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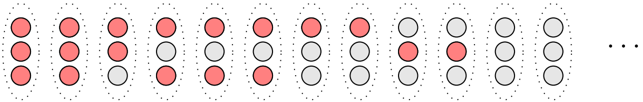

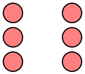

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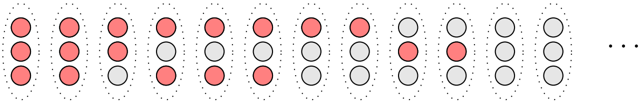

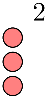

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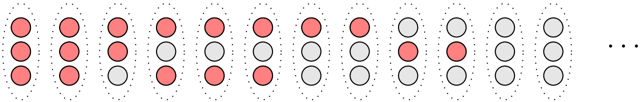

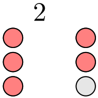

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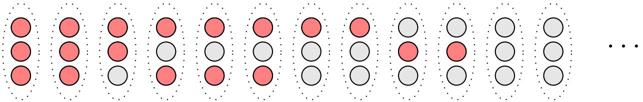

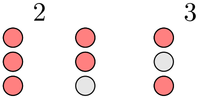

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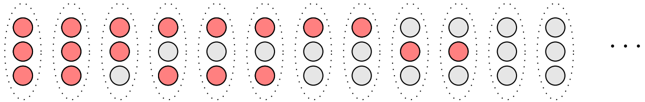

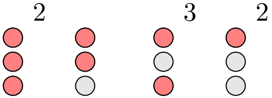

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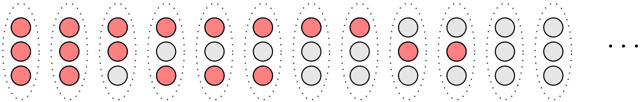

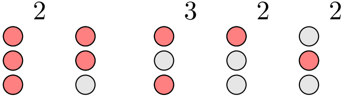

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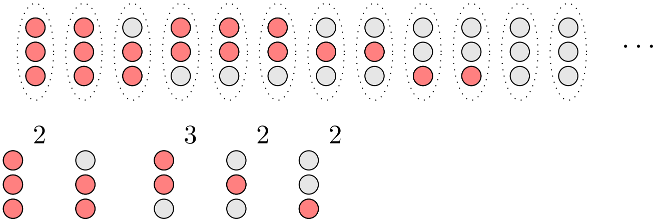

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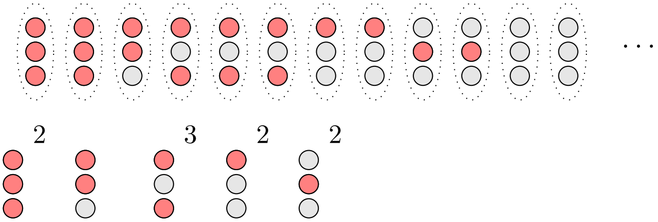

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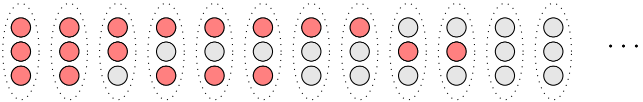

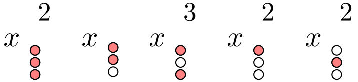

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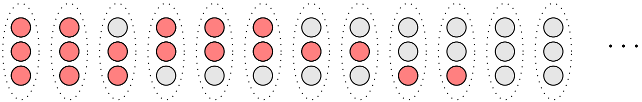

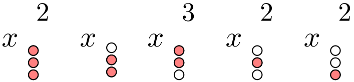

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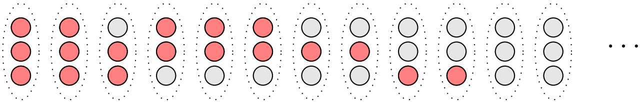

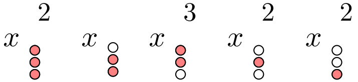

$\rightarrow$ C_3 acts on monomials

Direct product in the case of finite blocks

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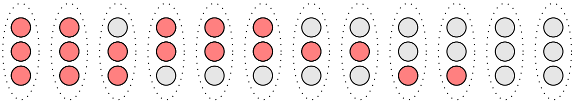


$\cdots$

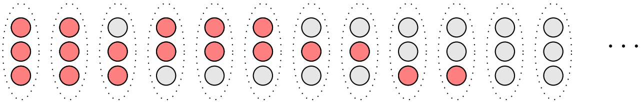
$G' = C_3$ *acting on (non empty) subsets*

$\mathbb{Q}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

Direct product in the case of finite blocks

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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x 

x 

Direct product in the case of finite blocks

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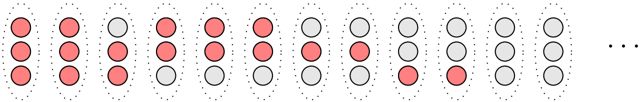

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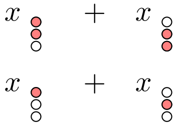

Direct product in the case of finite blocks

"Speak, friend..."

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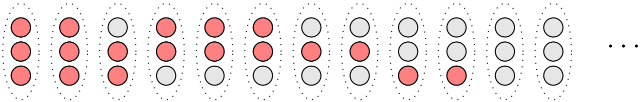

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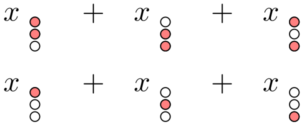

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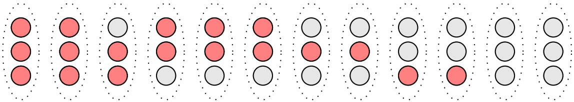 $\cdots$

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O(x )

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Direct product in the case of finite blocks

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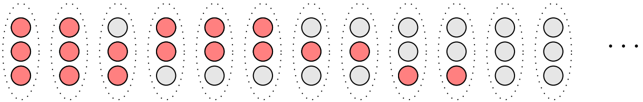

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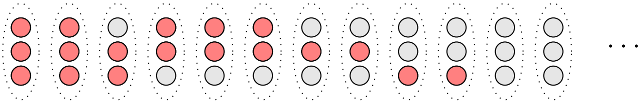

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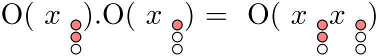

Direct product in the case of finite blocks

"Speak, friend..."

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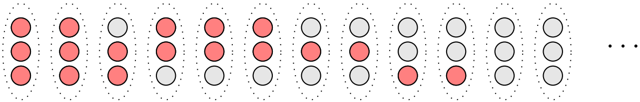

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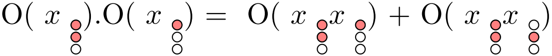

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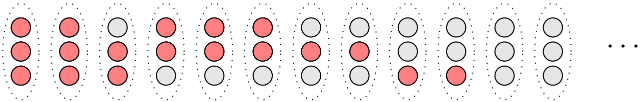

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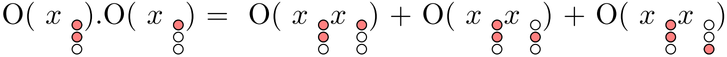

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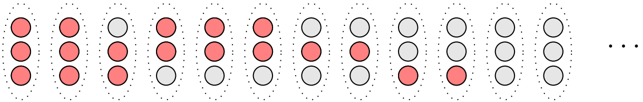

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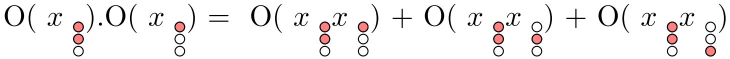

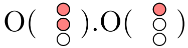

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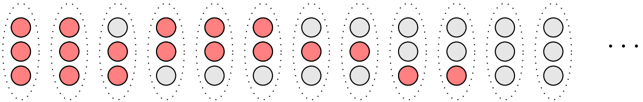

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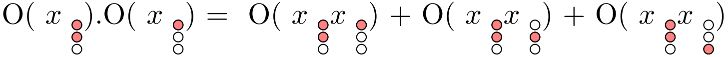

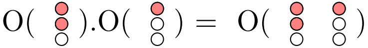

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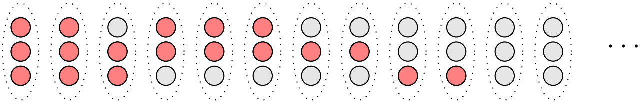

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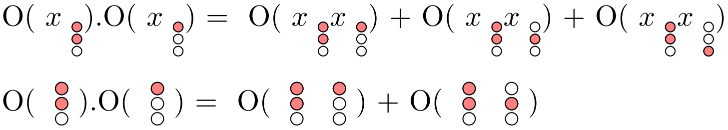

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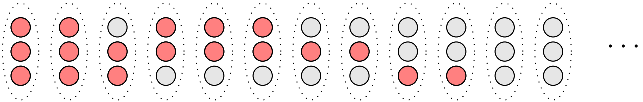

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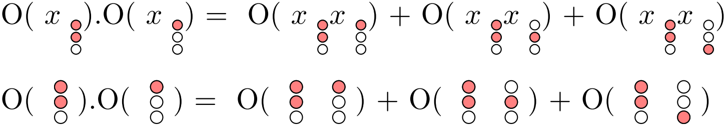

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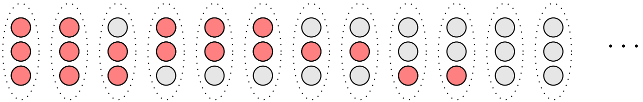

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