

Distributivity spectrum of forcing notions

Marlene Koelbing

joint work with Vera Fischer and Wolfgang Wohofsky

Universität Wien (Kurt Gödel Research Center)

`marlenekoelbing@web.de`

Models and Sets seminar, Leeds
(online via Zoom)

3 March 2021

Definition

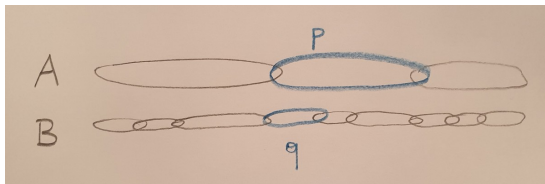
\mathbb{P} is **λ -distributive** if it does not add a function $f : \lambda \rightarrow \text{Ord}$ with $f \notin V$.
 $\mathfrak{h}(\mathbb{P}) :=$ least λ such that \mathbb{P} is **not** λ -distributive (the **distributivity** of \mathbb{P}).

Definition

\mathbb{P} is **λ -distributive** if it does not add a function $f : \lambda \rightarrow \text{Ord}$ with $f \notin V$.
 $\mathfrak{h}(\mathbb{P}) :=$ least λ such that \mathbb{P} is **not** λ -distributive (the **distributivity** of \mathbb{P}).

For maximal antichains A and B ,

$$B \text{ refines } A :\iff \forall q \in B \exists p \in A (q \leq p).$$



Proposition

\mathbb{P} is λ -distributive if and only if for each family $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ of maximal antichains in \mathbb{P} , there exists a common refinement (i.e., a maximal antichain B such that B refines A_ξ for each $\xi < \lambda$).

Definition

\mathbb{P} is **λ -distributive** if it does not add a function $f : \lambda \rightarrow \text{Ord}$ with $f \notin V$.
 $\mathfrak{h}(\mathbb{P}) :=$ least λ such that \mathbb{P} is **not** λ -distributive (the **distributivity** of \mathbb{P}).

Let $\lambda = \mathfrak{h}(\mathbb{P})$, and let $f : \lambda \rightarrow \text{Ord}$ witness this, i.e., $f \notin V$.

Note that $f \restriction \gamma \in V$ for every $\gamma < \lambda$ (f is not just **new**, but even **“fresh”**).

Definition (Fresh function spectrum)

We say that $\lambda \in \text{FRESH}(\mathbb{P})$ if in some extension of V by \mathbb{P} ,

there exists a **fresh function on λ** ,

i.e., a function $f : \lambda \rightarrow \text{Ord}$ with

- ① $f \notin V$, but
- ② $f \restriction \gamma \in V$ for every $\gamma < \lambda$.

Note: $\lambda \in \text{FRESH}(\mathbb{P}) \iff \text{cf}(\lambda) \in \text{FRESH}(\mathbb{P})$

So from now on, we only talk about **regular** cardinals λ .

- $\min(\text{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

Proposition

If $\lambda > |\mathbb{P}|$, then $\lambda \notin \text{FRESH}(\mathbb{P})$.

Proof (Sketch).

- assume towards contradiction that $\dot{f} : \lambda \rightarrow \text{Ord}$ is fresh
- for each $\gamma < \lambda$, fix $p_\gamma \in \mathbb{P}$ such that p_γ **decides** $\dot{f} \restriction \gamma$
- λ regular, so there exists p^* with $p_\gamma = p^*$ for **unboundedly** many γ
- so p^* decides \dot{f} (and hence \dot{f} is not new) □

- $\text{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]$

Example: Let \mathbb{C} be the usual ω -Cohen forcing (with $|\mathbb{C}| = \omega$).

$$\text{FRESH}(\mathbb{C}) = \{\omega\}$$

Let \mathbb{C}_μ be the forcing for adding μ many ω -Cohen reals (μ arbitrary).

$$\text{FRESH}(\mathbb{C}_\mu) = ?$$

Let \mathbb{C}_μ be the forcing for adding μ many ω -Cohen reals (μ arbitrary).

$$\text{FRESH}(\mathbb{C}_\mu) = \{\omega\}$$

Proposition

If \mathbb{P} satisfies $\mathbb{P} \times \mathbb{P}$ is δ -c.c. and $\lambda \geq \delta$, then $\lambda \notin \text{FRESH}(\mathbb{P})$.

Is \mathbb{P} being δ -c.c. sufficient? No: consider a Suslin tree T (on ω_1)

- T is c.c.c. (i.e., ω_1 -c.c.)
- **but** $\omega_1 \in \text{FRESH}(T)$
 - ▶ the generic branch is a fresh function on ω_1
- $\text{FRESH}(T) = \{\omega_1\}$, since...
 - ▶ T is ω -distributive
 - ▶ $|T| = \omega_1$

Proposition

If \mathbb{P} is δ -c.c. and $\lambda > \delta$, then $\lambda \notin \text{FRESH}(\mathbb{P})$.

Proposition

If \mathbb{P} **collapses** λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \text{FRESH}(\mathbb{P})$.

It follows that $[\mathfrak{h}(\mathbb{P}), \lambda] \subseteq \text{FRESH}(\mathbb{P})$.

Example (under CH): Let $\text{Coll}(\omega_1, \omega_2)$ be the forcing collapsing ω_2 to ω_1 with countable conditions.

$$\text{FRESH}(\text{Coll}(\omega_1, \omega_2)) = \{\omega_1, \omega_2\}$$

Theorem (Base Matrix Theorem; Balcar-Pelant-Simon)

$\mathcal{P}(\omega)/\text{fin}$ collapses \mathfrak{c} to \mathfrak{h} .

$$\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\text{fin})$$

Corollary

$$\text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}].$$

Corollary

$$\text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}].$$

Let \mathbb{M} denote **Mathias forcing**.

Proposition

$$\text{FRESH}(\mathbb{M}) = \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}].$$

Proof is based on $\mathbb{M} \cong \mathcal{P}(\omega)/\text{fin} * \mathbb{M}(G)$.

Possible fresh function spectra under GCH

Let A be a set of regular cardinals.

Does there exist a forcing \mathbb{P} such that $FRESH(\mathbb{P}) = A$?

Examples under GCH:

- $FRESH(\mathbb{C}(\lambda)) = \{\lambda\}$
 - ▶ $<\lambda$ -closed
 - ▶ $|\mathbb{C}(\lambda)| = \lambda$
- $FRESH(\mathbb{C} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1\}$
 - ▶ If a function is fresh, it **remains fresh** in any extension.
- $FRESH(\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_3)) = \{\omega_1, \omega_3\}$
 - ▶ $\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_3) \cong \mathbb{C}(\omega_3) * \check{\mathbb{C}}(\omega_1) \cong \mathbb{C}(\omega_3) * \mathbb{C}(\omega_1)$
- If A is finite: $FRESH(\prod_{\lambda \in A} \mathbb{C}(\lambda)) = A$

What if A is infinite?

Definition

A set A of regular cardinals is an **Easton set** if for every limit point α of A ,

- α regular $\Rightarrow \alpha \in A$,
- α singular $\Rightarrow \alpha^+ \in A$.

Let ${}^E\prod_{\lambda \in A} \mathbb{P}_\lambda$ denote the **Easton product** of the \mathbb{P}_λ :

- full support at singular limits,
- bounded support at regular limits (i.e., inaccessibles).

Theorem (GCH)

Let A be an Easton set. Then $\text{FRESH}({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda)) = A$.

If A is an Easton set, then there exists \mathbb{P} with $\text{FRESH}(\mathbb{P}) = A$.

Why Easton sets?

Example

Let α be the first inaccessible and $A \subseteq \alpha$ an unbounded subset of regular cardinals. Then ${}^E\prod_{\lambda \in A} \mathbb{C}(\lambda)$ adds an α -Cohen real.

${}^E\prod_{\lambda \in A} \mathbb{C}(\lambda)$ is the bounded support product and hence adds an α -Cohen real.

In particular, $\alpha \in \text{FRESH}({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

Example (GCH)

Let $A \subseteq \aleph_\omega$ be an unbounded subset of regular cardinals. Then ${}^E\prod_{\lambda \in A} \mathbb{C}(\lambda)$ adds an \aleph_ω^+ -Cohen real.

Follows from a paper of Shelah, which uses **pcf theory**.

In particular, $\aleph_\omega^+ \in \text{FRESH}({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda))$.

Are unbounded subsets $A \subseteq \aleph_\omega$ realizable?

Since $\aleph_{\omega+1} \notin A$, they are not Easton sets!

For $A = \{\aleph_n : n \in \omega\}$, yes:

- either $\text{Coll}(\aleph_0, \aleph_\omega)$ or finite support product of Cohen forcings work.

Question (GCH)

Let $A \subsetneq \aleph_\omega$ unbounded. Is there a forcing \mathbb{P} with $\text{FRESH}(\mathbb{P}) = A$?



Let α be regular limit, i.e, inaccessible.

Question (GCH)

Let $A \subseteq \alpha$ unbounded. Is there a forcing \mathbb{P} with $\text{FRESH}(\mathbb{P}) = A$?

Not even clear if A is the set of all regular cardinals strictly below α .

Omitting fresh function spectra

Let A be a set of regular cardinals.

Does there (always?) exist a forcing \mathbb{P} such that $FRESH(\mathbb{P}) = A$?

Remember:

$$FRESH(\mathbb{C}) = \{\omega\}$$

Is there a forcing \mathbb{P} with $FRESH(\mathbb{P}) = \{\omega_1\}$?

Yes, if:

- **CH holds** (then $\mathbb{C}(\omega_1)$ does the job),
- there **exists a Suslin tree** (then the Suslin tree does the job).

Is it provable in ZFC?

Definition

Todorćević's maximality principle is the following assertion:

If \mathbb{P} is a forcing which adds a fresh subset of ω_1 , then

- \mathbb{P} collapses ω_1 , or
- \mathbb{P} collapses ω_2 .

It is consistent, relative to the existence of an inaccessible cardinal.

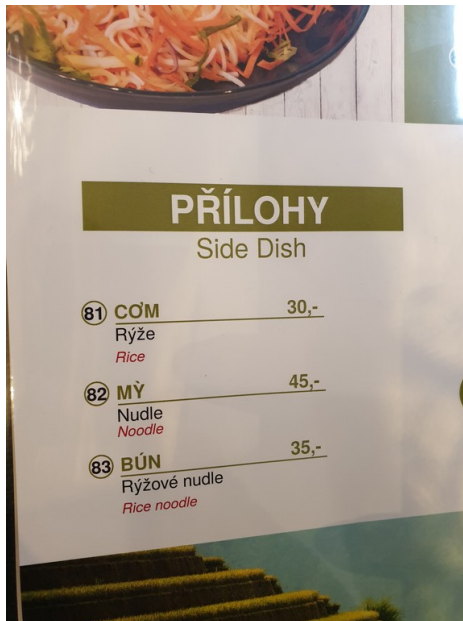
Theorem

Assume Todorćević's maximality principle, and “ $0^\#$ does not exist”.

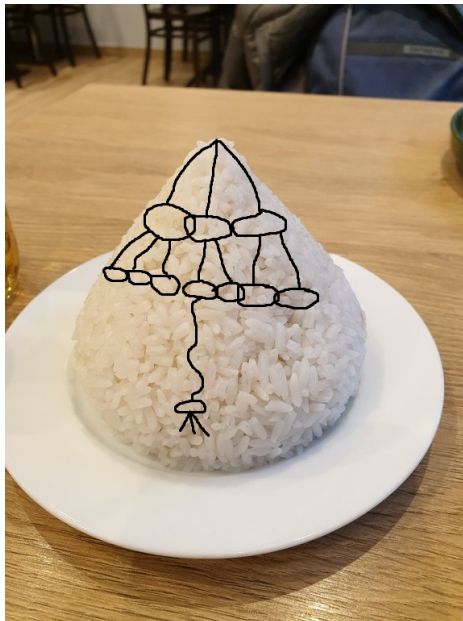
Then for every forcing with $\omega_1 \in \text{FRESH}(\mathbb{P})$,

- $\omega \in \text{FRESH}(\mathbb{P})$, or
- $\omega_2 \in \text{FRESH}(\mathbb{P})$.

In particular, there is no forcing \mathbb{P} with $\text{FRESH}(\mathbb{P}) = \{\omega_1\}$.







Remember:

Proposition

\mathbb{P} is λ -distributive if and only if for each family $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ of maximal antichains in \mathbb{P} , there exists a common refinement (i.e., a maximal antichain B such that B refines A_ξ for each $\xi < \lambda$).



Definition (Combinatorial distributivity spectrum)

We say $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ is a **distributivity matrix of height λ (for \mathbb{P})** if

- A_ξ is a **maximal antichain** in \mathbb{P} (for each $\xi < \lambda$),
- A_η **refines** A_ξ whenever $\eta \geq \xi$,
 - ▶ A_η refines $A_\xi : \iff \forall q \in A_\eta \exists p \in A_\xi (q \leq p)$
- there is **no common refinement**
 - ▶ i.e., there is no maximal antichain B which refines every A_ξ .

We say $\lambda \in \text{COM}(\mathbb{P})$ if there is a distributivity matrix of height λ for \mathbb{P} .



Is $\text{COM}(\mathbb{P}) = \text{FRESH}(\mathbb{P})$?

Proposition

$$\min(\text{COM}(\mathbb{P})) = \min(\text{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$$

Proposition

$$\text{COM}(\mathbb{P}) \subseteq \text{FRESH}(\mathbb{P})$$

Proposition

$\text{COM}(\mathbb{P}) = \text{FRESH}(\mathbb{P})$ in case \mathbb{P} is a **complete** Boolean Algebra

But note:

The Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is **not complete!!**

$$\{\mathfrak{h}\} \subseteq \text{COM}(\mathcal{P}(\omega)/\text{fin}) \subseteq \text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$$

Theorem

Let V_0 be a model of ZFC which satisfies GCH. In V_0 , let

$$\omega_1 < \lambda \leq \mu$$

be cardinals such that λ is regular and $\text{cf}(\mu) > \omega$. Then there is a c.c.c. (and hence cofinality preserving) extension W of V_0 such that

$$W \models \omega_1 = \mathfrak{h} = \mathfrak{b} \wedge \lambda \in \text{COM}(\mathcal{P}(\omega)/\text{fin}) \wedge \mu = \mathfrak{c}.$$

- ① go to the usual Cohen model with $\mu = \mathfrak{c}$
- ② forcing (iteration) to add a distributivity matrix of height λ
- ③ show that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ in the final model

Corollary

It is consistent that $\mathfrak{h} < \mathfrak{c} = \omega_2$, and

$$\{\omega_1, \omega_2\} = \text{COM}(\mathcal{P}(\omega)/\text{fin}) = \text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$$

Our definition is inspired by Hechler's forcings defined in "*Short complete nested sequences in $\beta\mathbb{N}\setminus\mathbb{N}$ and small maximal almost-disjoint families*".

Definition

Let $T := \{\sigma \in \lambda^{<\lambda} : \sigma \text{ has successor length}\}$.

Define a forcing \mathbb{Q} as follows: p is a condition if

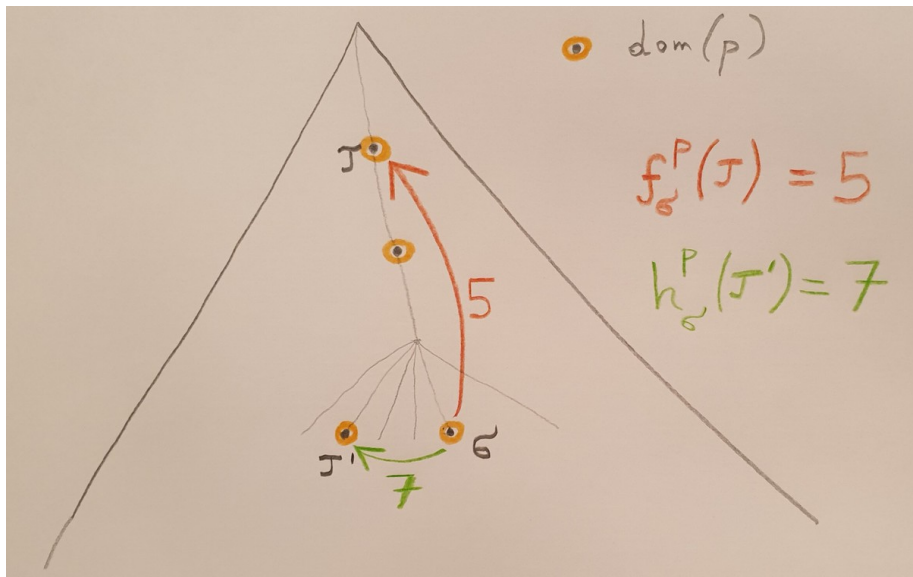
- p is a finite function with $\text{dom}(p) \subseteq T$,
- for each $\sigma \in \text{dom}(p)$, $p(\sigma) = (s_\sigma^p, f_\sigma^p, h_\sigma^p)$, with $s_\sigma^p \in 2^{<\omega}$.

If G is a generic filter, let

- $a_\sigma := \bigcup_{p \in G} s_\sigma^p$ (which is an infinite subset of ω),
- $A_{\xi+1} := \{a_\sigma : \sigma \text{ has length } \xi + 1\}$ (for $\xi < \lambda$),
- $\mathcal{A} := \{A_{\xi+1} : \xi < \lambda\}$ is the intended generic distributivity matrix.

f_σ^p : promises for the \subseteq^* -relation

h_σ^p : promises for the almost disjointness

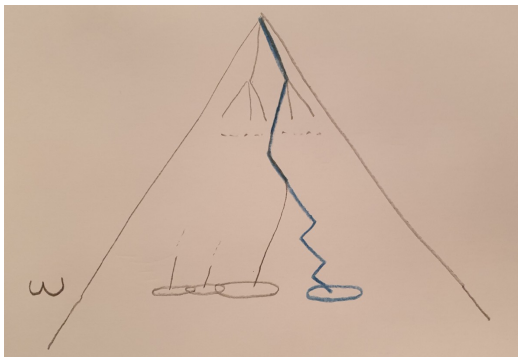


Problem

The levels of the generic matrix are **not mad** families (new reals are added).

Problem

The levels of the generic matrix are **not mad** families (new reals are added).



Solution

Use iterated forcing to make sure that the levels are maximal in the end.

- finite support iteration of length λ

A distributivity matrix of height λ is added

Finite support iteration of length λ yields distributivity matrix of height λ .

Main steps of the proof:

- iterands have ccc
- each node of $\lambda^{<\lambda}$ appears at some intermediate stage of the iteration
 - ▶ so in the final model, a_σ is defined for all $\sigma \in \lambda^{<\lambda}$
- along branches through $\lambda^{<\lambda}$, we have \subseteq^* -**decreasing** sequences
- on levels of $\lambda^{<\lambda}$, we have **almost disjoint** families
- we have **towers** along branches (so no elements intersect the matrix)
- we have **mad** families on levels

Maximality: if $b \subseteq \omega$ is a challenge, use a complete subforcing to capture b ; then find a $\sigma \in \lambda^{<\lambda}$ which is “outside” the complete subforcing, and use genericity to show that b and a_σ do have infinite intersection.

Showing $\omega_1 = \mathfrak{h} = \mathfrak{b}$

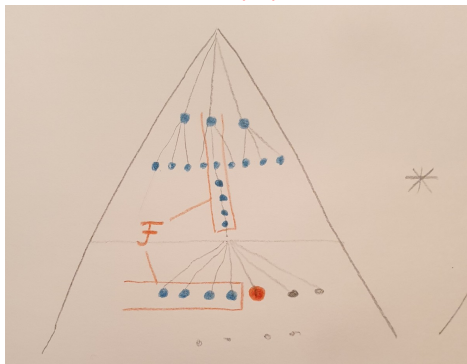
Main ideas of the proof:

- recall that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$
- so it is enough to show that $\mathfrak{b} = \omega_1$
- we show that the ground model reals $\mathcal{B} := \omega^\omega \cap V_0$ remain unbounded

Showing $\omega_1 = \mathfrak{h} = \mathfrak{b}$

Main ideas of the proof:

- recall that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$
- so it is enough to show that $\mathfrak{b} = \omega_1$
- we show that the ground model reals $\mathcal{B} := \omega^\omega \cap V_0$ **remain unbounded**
- we represent our iteration as a “finer” iteration whose iterands are equivalent to **Mathias forcing $\mathbb{M}(\mathcal{F})$ with respect to a filter \mathcal{F}**



Showing $\omega_1 = \mathfrak{h} = \mathfrak{b}$

Main ideas of the proof:

- recall that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$
- so it is enough to show that $\mathfrak{b} = \omega_1$
- we show that the ground model reals $\mathcal{B} := \omega^\omega \cap V_0$ remain unbounded
- we represent our iteration as a “finer” iteration whose iterands are equivalent to **Mathias forcing $\mathbb{M}(\mathcal{F})$ with respect to a filter \mathcal{F}**
- we show that the filters \mathcal{F} are \mathcal{B} -Canjar
 - ▶ a filter \mathcal{F} is **\mathcal{B} -Canjar**, if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of \mathcal{B}
 - ▶ we use a combinatorial characterization by Guzmán-Hrušák-Martínez (extending a characterization of Canjarness given by Hrušák-Minami)
- at limit steps, the unboundedness of \mathcal{B} is preserved by a theorem of Judah-Shelah

Thank you for your attention and greetings from Vienna...



Augarten, 3rd December 2020

Thank you for your attention and greetings from Vienna...



Stephansplatz (first district) during first lockdown, 9th April 2020

Thank you for your attention and greetings from Vienna...



Graben (first district) during first lockdown, 9th April 2020