

Arithmetic under negated induction

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[T]he method is extremely valuable when we want to beat a particular theory into the ground. When it can be carried out, the method of elimination of quantifiers gives a tremendous amount of information about a theory.

Chang–Keisler 1973

This talk

Main theorem

The theory $WKL_0^* + \neg I\Sigma_1^0$ in the language for second-order arithmetic eliminates set quantifiers.

Plan

1. induction and collection
2. the Weak König Lemma
3. quantifier elimination
4. consequences

[T]he method is extremely valuable when we want to beat a particular theory into the ground. When it can be carried out, the method of elimination of quantifiers gives a tremendous amount of information about a theory.

Chang–Keisler 1973

First-order arithmetic

- ▶ $\mathcal{L}_1 = \{0, 1, +, \times, <, =\}$.
- ▶ A quantifier is *bounded* if it is of the form $\forall v < t$ or $\exists v < t$.
- ▶ An \mathcal{L}_1 formula is Δ_0 if all its quantifiers are bounded.
- ▶ $\Sigma_n = \{\exists \bar{v}_1 \forall \bar{v}_2 \cdots Q \bar{v}_n \theta : \theta \in \Delta_0\}$ and $\Pi_n = \{\forall \bar{v}_1 \exists \bar{v}_2 \cdots Q' \bar{v}_n \theta : \theta \in \Delta_0\}$.
- ▶ $I\Sigma_n$ consists of the axioms of PA^- and for every $\theta \in \Sigma_n$,

$$\theta(0) \wedge \forall x (\theta(x) \rightarrow \theta(x+1)) \rightarrow \forall x \theta(x).$$

- ▶ $PA = \bigcup_{k \in \mathbb{N}} I\Sigma_k$.
- ▶ exp asserts the totality of $x \mapsto 2^x$ over $I\Sigma_0$.
- ▶ $B\Sigma_n$ consists of the axioms of $I\Sigma_0$ and for every $\theta \in \Sigma_n$,

$$\forall a (\forall x < a \exists y \theta(x, y) \rightarrow \exists b \forall x < a \exists y < b \theta(x, y)).$$

Theorem (Paris–Kirby 1978)

$I\Sigma_0 + \text{exp} \dashv\vdash B\Sigma_1 + \text{exp} \dashv\vdash I\Sigma_1 \dashv\vdash B\Sigma_2 \dashv\vdash I\Sigma_2 \dashv\vdash B\Sigma_3 \dashv\vdash I\Sigma_3 \dashv\vdash B\Sigma_4 \dashv\vdash I\Sigma_4 \dashv\vdash \cdots$ and none of the converses holds.

Model theory of fragments of PA

 $n \in \mathbb{N}$

Insight (Kaye, around 1991)

The model-theoretic properties of a model of arithmetic do not only depend on the induction axioms it satisfies, but also on the induction axioms it does *not* satisfy.

Theorem (Kossak 1990, Kaye 1991)

Every countable model of $B\Sigma_{n+1} + \text{exp} + \neg I\Sigma_{n+1}$ has 2^{\aleph_0} -many automorphisms and proper elementary cofinal substructures.

Theorem (Paris–Kirby 1978)

There is a countable model of $I\Sigma_n + \text{exp} + \neg B\Sigma_{n+1}$ with no non-trivial automorphism and no proper elementary substructure.

Theorem (Paris–Kirby 1978)

$I\Sigma_0 + \text{exp} \not\vdash B\Sigma_1 + \text{exp} \dashv\vdash I\Sigma_1 \dashv\vdash B\Sigma_2 \dashv\vdash I\Sigma_2 \dashv\vdash B\Sigma_3 \dashv\vdash I\Sigma_3 \dashv\vdash B\Sigma_4 \dashv\vdash I\Sigma_4 \dashv\vdash \dots$ and none of the converses holds.

Second-order arithmetic

- ▶ $\mathcal{L}_2 := \mathcal{L}_1 \cup \{\in\}$ has a number sort and a set sort.
- ▶ We use lowercase letters a, b, c, \dots, x, y, z for objects of the number sort, and uppercase letters A, B, C, \dots, X, Y, Z for objects of the set sort.
- ▶ We write an \mathcal{L}_2 structure as (M, \mathcal{X}) , where M, \mathcal{X} are universes for the number and set sorts respectively.
- ▶ A quantifier is *bounded* if it is of the form $\forall v < t$ or $\exists v < t$.
- ▶ An \mathcal{L}_2 formula is Δ_0^0 if all its quantifiers are bounded.
- ▶ $\Sigma_n^0 = \{\exists \bar{v}_1 \forall \bar{v}_2 \dots Q \bar{v}_n \theta : \theta \in \Delta_0^0\}$ and $\Pi_n^0 = \{\forall \bar{v}_1 \exists \bar{v}_2 \dots Q' \bar{v}_n \theta : \theta \in \Delta_0^0\}$.
- ▶ An \mathcal{L}_2 formula is Δ_0^1 or *arithmetical* if all its quantifiers are of the number sort.
- ▶ $\Sigma_n^1 = \{\exists \bar{V}_1 \forall \bar{V}_2 \dots Q \bar{V}_n \theta : \theta \in \Delta_0^1\}$ and $\Pi_n^1 = \{\forall \bar{V}_1 \exists \bar{V}_2 \dots Q' \bar{V}_n \theta : \theta \in \Delta_0^1\}$.
- ▶ IS_n^0 and BS_n^0 are similar to IS_n and BS_n , except that there may be set variables.

Theorem (Paris–Kirby 1978)

$\text{IS}_0^0 + \text{exp} \dashv \vdash \text{BS}_1^0 + \text{exp} \dashv \vdash \text{IS}_1^0 \dashv \vdash \text{BS}_2^0 \dashv \vdash \text{IS}_2^0 \dashv \vdash \text{BS}_3^0 \dashv \vdash \text{IS}_3^0 \dashv \vdash \text{BS}_4^0 \dashv \vdash \text{IS}_4^0 \dashv \vdash \dots$ and none of the converses holds.

ω -extensions

Definition

An ω -extension of an \mathcal{L}_2 structure is an extension with no new number.

Theorem (Towsner 2015 for $n \geq 1$)

Given any countable $(M, \mathcal{X}) \models \text{IS}_n^0 + \text{exp} + \neg \text{BS}_{n+1}^0$ and any $S \subseteq M$, one can ω -extend (M, \mathcal{X}) to $(M, \mathcal{Y}) \models \text{IS}_n^0 + \text{exp} + \neg \text{BS}_{n+1}^0$ in which S is definable.

Proposition

For every countable $(M, \mathcal{X}) \models \text{BS}_{n+1}^0 + \text{exp} + \neg \text{IS}_{n+1}^0$, there is $S \subseteq M$ such that one can *never* ω -extend (M, \mathcal{X}) to $(M, \mathcal{Y}) \models \text{BS}_{n+1}^0 + \text{exp} + \neg \text{IS}_{n+1}^0$ in which S is definable.

Theorem (Paris–Kirby 1978)

$\text{IS}_0^0 + \text{exp} \dashv \vdash \text{BS}_1^0 + \text{exp} \dashv \vdash \text{IS}_1^0 \dashv \vdash \text{BS}_2^0 \dashv \vdash \text{IS}_2^0 \dashv \vdash \text{BS}_3^0 \dashv \vdash \text{IS}_3^0 \dashv \vdash \text{BS}_4^0 \dashv \vdash \text{IS}_4^0 \dashv \vdash \dots$ and none of the converses holds.

Preservation theorem for Π_1^1 formulas

Definition

An *ω -extension* of an \mathcal{L}_2 structure is an extension with no new number.

Lemma (elementary)

Let T, T^* be \mathcal{L}_2 theories. If every countable model of T has an ω -extension to a model of T^* , then $T \vdash \Pi_1^1\text{-Th}(T^*)$.

We identify \mathcal{L}_2 formulas with their Gödel numbers.

Definition

A type $p(\bar{v}, \bar{V})$ over an \mathcal{L}_2 structure (M, \mathcal{X}) is *recursive* if it involves only finitely many free variables and finitely many parameters $\bar{c}, \bar{C} \in (M, \mathcal{X})$, and

$$\{\theta(\bar{v}, \bar{V}, \bar{z}, \bar{Z}) : \theta(\bar{v}, \bar{V}, \bar{c}, \bar{C}) \in p(\bar{v}, \bar{V})\}$$

is recursive. The structure is *recursively saturated* if it realizes all recursive types.

Theorem (Barwise 1975, Ressayre 1977, independently)

Let T, T^* be \mathcal{L}_2 theories. If $T \vdash \Pi_1^1\text{-Th}(T^*)$, then every countable *recursively saturated* model of T has an ω -extension to a model of T^* .

Ramsey's Theorem for pairs for two colours

Definition

RT_2^2 denotes an \mathcal{L}_2 sentence which expresses “whenever each unordered pair of numbers is given exactly one of two colours, there is an unbounded monochromatic set” over $I\Sigma_0 + \text{exp}$.

Open question

Does $B\Sigma_2^0 \vdash \Pi_1^1\text{-Th}(RCA_0 + RT_2^2)$?

$RCA_0 = I\Sigma_1^0 + \Delta_1^0\text{-comprehension}$.

Model-theoretic version of the question

Does every countable recursively saturated $(M, \mathcal{X}) \models B\Sigma_2^0$ have an ω -extension to a model of $RCA_0 + RT_2^2$?

Partial answer (Cholak–Jockusch–Slaman 2001)

Yes, if $(M, \mathcal{X}) \models I\Sigma_2^0$.

Remaining question

Does every countable recursively saturated $(M, \mathcal{X}) \models B\Sigma_2^0 + \neg I\Sigma_2^0$ have an ω -extension to a model of $RCA_0 + RT_2^2$?

The Weak König Lemma

- ▶ $B\Sigma_1 + \text{exp}$ and WKL_0^* are respectively the first- and the second-order theories of exponentially closed initial segments in models of arithmetic.
- ▶ WKL_0^* consists of $I\Sigma_0^0 + \text{exp}$, the Δ_1^0 comprehension scheme, and an axiom stating “every unbounded 0–1 tree has an unbounded path”.

Proposition (Simpson–Smith 1986)

$WKL_0^* \vdash B\Sigma_1^0 + \text{exp}$.

Any set of numbers that is both Σ_1^0 - and Π_1^0 -definable is in the set universe.

Theorem (Simpson–Smith 1986)

Every countable model of $B\Sigma_1^0 + \text{exp}$ has an ω -extension to a model of WKL_0^* .

Proof

Force in the style of Jockusch–Soare (1972), where conditions are unbounded trees. \square

Corollary (Simpson–Smith 1986)

$B\Sigma_1^0 + \text{exp}$ axiomatizes $\Pi_1^1\text{-Th}(WKL_0^*)$.

The model-theoretic core

Theorem

Every countable $(M, \mathcal{X}) \models \text{BS}\Sigma_1^0 + \text{exp} + \neg \text{IS}\Sigma_1^0$ has a unique countable ω -extension $(M, \mathcal{Y}) \models \text{WKL}_0^* + \neg \text{IS}\Sigma_1^0$ up to isomorphism.

Proof sketch

Given two such extensions $(M, \mathcal{Y}), (M, \mathcal{Z})$, build an isomorphism between them by a back-and-forth construction.

At every stage, we have $\bar{r}, \bar{R} \in (M, \mathcal{Y})$ and $\bar{s}, \bar{S} \in (M, \mathcal{Z})$ such that

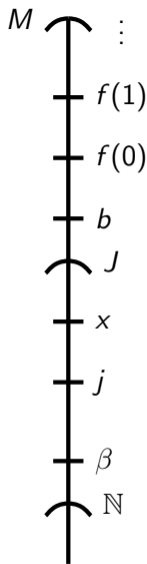
$$\exists \beta \in M \setminus \mathbb{N} \quad \exists b \in M \setminus J \quad \forall \Sigma_0^0 \text{ formula } \theta < \beta \quad \forall \bar{x} < b \quad \forall \bar{j} \in J \\ (M, \mathcal{Y}) \models \theta(\bar{x}, f(\bar{j}), \bar{r}, \bar{R}, A) \Leftrightarrow (M, \mathcal{Z}) \models \theta(\bar{x}, f(\bar{j}), \bar{s}, \bar{S}, A),$$

where

- ▶ J is a proper initial segment of M that is closed under $x \mapsto 2^x$ and is Σ_1^0 -definable over the parameter $A \in \mathcal{X}$ in (M, \mathcal{X}) ; and
- ▶ $f: J \rightarrow M$ whose graph is Σ_0^0 -definable over A in (M, \mathcal{X}) and whose range is cofinal in M . □

$$\forall b \exists w < a \forall v < b \dots \\ \rightarrow \exists w < a \forall v \dots$$

$$\forall b \exists S \forall v < b \dots \rightarrow \exists S \forall v \dots$$



Quantifier elimination

Theorem

Every countable $(M, \mathcal{X}) \models \text{BS}\Sigma_1^0 + \text{exp} + \neg\text{IS}\Sigma_1^0$ has a unique countable ω -extension $(M, \mathcal{Y}) \models \text{WKL}_0^* + \neg\text{IS}\Sigma_1^0$ up to isomorphism.

Lemma (folklore?)

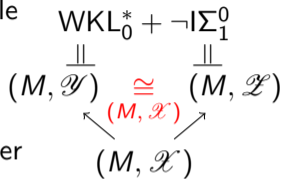
A theory T has quantifier elimination if the following is true: whenever A is a common substructure of $M, N \models T$, if $\bar{a} \in A$ such that $M \models \exists y \varphi(\bar{a}, y)$, where φ is quantifier-free, then $N \models \exists y \varphi(\bar{a}, y)$.

Main theorem

Every \mathcal{L}_2 formula is equivalent to a Δ_1^0 formula over $\text{WKL}_0^* + \neg\text{IS}\Sigma_1^0$.

Proof

Run a proof of the lemma above on countable recursively saturated models, so that all extensions can be assumed to be ω -extensions. \square



Over $\text{WKL}_0^* + \neg\text{IS}\Sigma_1^0$, every \mathcal{L}_2 formula $\theta(\bar{x}, \bar{Y})$ is equivalent to $\neg\text{IS}\Sigma_1^Z \wedge \mathbb{J}\mathbb{S}_{\bar{Y}, Z} \Vdash \theta(\bar{x}, \bar{Y})$.

The Weak König Lemma as a model completion

Main theorem

Every \mathcal{L}_2 formula is equivalent to a Δ_0^1 formula over $\text{WKL}_0^* + \neg I\Sigma_1^0$.

Corollary

$\text{WKL}_0^* + \neg I\Sigma_1^0$ is the unique \mathcal{L}_2 theory T such that

- (a) $\Pi_1^1\text{-Th}(T) = \Pi_1^1\text{-Th}(\text{B}\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0)$; and
- (b) every Π_1^1 formula is equivalent to a Σ_1^1 formula over T .

$\Delta_0^1 \mapsto$ quantifier-free

$\Pi_1^1 \mapsto \forall_1$

$\Pi_2^1 \mapsto \forall_2$

Theorem (Simpson 1999, after Kleene)

Provably in ACA_0 , some Π_1^1 formula is not equivalent to any Σ_1^1 formula. In particular, this fact is true in $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Corollary

Models of $\text{WKL}_0^* + \neg I\Sigma_1^0$ are precisely the Σ_1^1 -closed models of $\text{B}\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$, i.e., if a Σ_1^1 formula, possibly with parameters, can be satisfied in a Δ_0^1 -elementary extension satisfying $\text{B}\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$, then it is already true in the ground model.

Π_1^1 conservativity over $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$

Corollary

The following are equivalent for all Π_2^1 sentences σ .

- (i) $\Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0 + \sigma) = \Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0)$.
- (ii) $\text{WKL}_0^* + \neg I\Sigma_1^0 \vdash \sigma$.
- (iii) Every countable model of $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$ has an ω -extension satisfying $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0 + \sigma$.

σ is Π_1^1 -conservative over $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$.

WKL_0^* is the strongest Π_1^1 -conservative Π_2^1 sentence over $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$

Theorem

$\text{WKL}_0^* + \neg I\Sigma_1^0 \not\vdash \text{RT}_2^2$.

Turing-equivalent to $\Pi_2\text{-Th}(\mathbb{N})$

Theorem

$\{\sigma \in \Pi_2^1\text{-Snt} : \Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp} + \sigma) = \Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp})\}$ is Π_2 -complete.

Proposition

There is a Π_2^1 sentence σ such that $\Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp} + \sigma) = \Pi_1^1\text{-Th}(B\Sigma_1^0 + \text{exp})$ but some countable model of $B\Sigma_1^0 + \text{exp}$ does not ω -extend to any model of $B\Sigma_1^0 + \text{exp} + \sigma$.

The model theory of $B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$

Main theorem

Every \mathcal{L}_2 formula is equivalent to a Δ_0^1 formula over $WKL_0^* + \neg I\Sigma_1^0$.

Model-theoretic core

Every countable $(M, \mathcal{X}) \models B\Sigma_1^0 + \text{exp} + \neg I\Sigma_1^0$ has a unique countable ω -extension $(M, \mathcal{Y}) \models WKL_0^* + \neg I\Sigma_1^0$ up to isomorphism.

WKL_0^* is the strongest Π_1^1 -conservative Π_2^1 sentence.

- ▶ These results relativize to $B\Sigma_{n+1}^0 + \text{exp} + \neg I\Sigma_{n+1}^0$.
- ▶ The model theory of $I\Sigma_n^0 + \text{exp} + \neg B\Sigma_{n+1}^0$ is radically different.
- ▶ Such model-theoretic properties cannot be achieved without including a false-in- \mathbb{N} sentence in the theory.

Insight (Kaye, around 1991)

The model-theoretic properties of a model of arithmetic do not only depend on the induction axioms it satisfies, but also on the induction axioms it does *not* satisfy.

\vdots
 \top
 $I\Sigma_2^0$
 \top
 $B\Sigma_2^0$
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 $I\Sigma_1^0$
 \top
 $B\Sigma_1^0 + \text{exp}$
 \top
 $I\Sigma_0^0 + \text{exp}$