Good Afternoon!
on Wednesday January 13th
2021!

Seminar Models and Sets
Leeds UK.

Strongly NIP almost real closed fields. Joint work with S. Krapp and G. Lehericy.

to appear in Math. Logic Quarterly

Plan of the talk:

1. Motivation 2. main results.
3. sketch proof for the main theorem
   highlight "reduction to Hahn fields via elementary equivalence.

1.1. Shelah's conjecture (reformulated in Dupont + Hasson Arch. Math. Logic 58 (2019)

Conjecture: Let K be an infinite strongly NIP field. Then either (1) K is a real closed field or

(2) K is algebraically closed or

(3) K admits a non-trivial $\mathcal{L}_r$-definable henselian valuation.

Here $\mathcal{L}_r$: the language of rings.

NIP: Not the Independence Property

- Let $\mathcal{L}$ language, T an $\mathcal{L}$-theory.
- Fix $\mathcal{M} \models T$ monster model
- Let $\varphi(\underline{x}\,;\,\underline{y})$ be an $\mathcal{L}$-formula.
- We say that φ has the IP if there are $(\underline{a}_i\,;\, i \in \mathbb{N})$ and $(\underline{b}_I\,;\, I \in \mathcal{P}(\mathbb{N}))$ s.t.

$\forall\, i \in \mathbb{N}$ and $I \in \mathcal{P}(\mathbb{N})$:

$i \in I$ iff $\mathcal{U} \models \varphi(\underline{a}_i; \underline{b}_I)$

- We say T has IP if there is such a φ.
- Otherwise we say T is NIP.
- If $\mathcal{N}$ structure, $\mathcal{N}$ is NIP if $Th(\mathcal{N})$ is NIP.

- Let $A \subseteq M$ set of parameters
  $\Delta \subseteq \mathcal{L}$- Formulas.
  $(J, <)$ linearly ord index set.
  $S = (a_j \mid j \in J)$ in M.

We say S is Δ- indiscernible over A if for every $k \in \mathbb{N}$ and

$i_1 < \dots < i_k$ and $j_1 < \dots < j_k$ in J and

$\varphi(x_1, \dots, x_k; \underline{y}) \in \Delta$ and $\underline{b} \in A$:

$\mathcal{U} \models \varphi(a_{i_1}, \dots, a_{i_k}; \underline{b}) \leftrightarrow \varphi(a_{j_1}, \dots a_{j_k}; \underline{b})$

- a family $(S_t \;;\; t \in X)$ in M is mutually indisc. over A if for every $u \in X$ then

$$S_u \text{ is indisc. over } A \cup \bigcup_{\substack{t \in X \\ t \neq u}} S_t$$

- Let p be a partial n-type over $A \subseteq M$. and let κ be a cardinal.

dp-rank of p over A:

$\text{dp-rk}(p, A) < \kappa$ if for every $(S_t \mid t < \kappa)$ of mutually indisc. sequences over A and $\underline{b} \in M^n$ realising p in $\mathcal{M}$, there is a $t < \kappa$ s.t. S_t is indisc. over $A \cup \{b_1, \dots, b_n\}$

- T is <u>strongly NIP</u> if T is NIP and $\text{dp-rk}(\{x = x\} ; \emptyset) < \aleph_0$.
- T is <u>dp-minimal</u> if T is NIP and $\text{dp-rk}(\{x=x\}; \emptyset) = 1$.

- real closed field : K is tot. ordered and $K[\sqrt{-1}]$ = alg closure $\tilde{K}$
- a valuation on a field K is a map $v : K \longrightarrow G \cup \{\infty\}$ where G is a t.o.a.g, $\infty > G$

s.t
$$v(a) = \infty \text{ iff } a = 0$$
$$v(ab) = v(a) + v(b)$$
$$v(a+b) \geq \min\{v(a), v(b)\}.$$

- to v associate $\mathcal{O}_v := \{x \in K ; v(x) \geq 0\}$ valuation ring is a local ring with a unique max ideal m_v ; $Kv := \mathcal{O}_v / m_v$ is the residue field, $vK = G$ is the value gp.
- v is definable iff $\mathcal{O}_v$ is definable
- v is henselian if $p(x) = x^n + x^{n-1} + a_{n-2}x^{n-2} + \dots + a_0$ where $a_{n-2}, .., a_0 \in m_v$

has a root in K.

2. Results. ORDERED FIELDS.

2.1: Specialize to $(K,<)$:

Conjecture 1.1: Let $(K,<)$ be strongly NIP ordered field. Then either $(K,<)$ is real closed or admits a non-trivial $\mathcal{L}_{or}$-definable henselian valuation.

strict dichotomy!

Note: Henselian $\Rightarrow$ Convex.

- Examples of strongly NIP ordered fields:

o-minimal $\xrightarrow{\text{FACT!}}$ weakly o-minimal $\longrightarrow$ dp-minimal $\longrightarrow$ strongly NIP $\longrightarrow$ NIP.

2.2. Specialise to almost real closed fields.

<u>Definition</u>: Let $(K, <)$ be an ord. field, assume that $v : K \twoheadrightarrow G$ a henselian valuation on K s.t Kv is real closed with value group G.

We say $(K, <)$ is a. r.c. w.r.t v and G. Delon + Farré.

Precursor: Becker + Gondard

<u>Examples</u>: Hahn Fields on G OAG

$$\mathbb{R}((G)) = \{ s = \sum s_g t^g \mid s_g \in \mathbb{R}; \text{ support } s \subset G \text{ is a well ord. subset} \}$$

For any OAG $\mathbb{R}((G))$ is almost real closed w.r.t v_{min} and G

where $v_{min}(s) := \min \text{support } s$

$v_{min}(0) = \infty$. v_{min} Henselian

Conjecture 1.2: Any strongly NIP ordered field $(K, <)$ is almost real closed.

Theorem 1.3: Conjecture 1.1 and Conjecture 1.2 are equivalent!

2.3 Exkurs: Results on dp-minimal ord. fields.

- Conjecture 1.1 has been verified for dp-minimal ord fields by Jahnke.

For later use:

Fact 4.6: [Halevi Hasson] TFAE G OAG.

(1) G is strongly NIP

(2) $G \equiv \bigoplus_{i \in I} G_i$ s.t.: $\forall p$ prime

$|\{i \in I \,;\, pG_i \neq G_i\}| < \infty$ and $\forall i \in I$

$|\{p \text{ prime} \,;\, [G_i : pG_i] = \infty\}| < \infty$.

[Conjecture 1.2 is also verified for dp-minimal ordered fields!]

Examples: again via Hahn Construction.

$\mathbb{R}((G))$ $\quad$ G dp-OAG.

Proposition 4.4: Let $(K,<)$ be an ord. fd. Then $(K,<)$ is dp-minimal iff it is almost real closed w.r.t some dp-minimal ordered abelian group.

2.4. Complete Characterization of strongly NIP almost RCF:

Question 1.4 Is it true that an ord field $(K,<)$ is strongly NIP if and only if it is almost real closed w.r.t a strongly NIP G?

Our main result is:

Theorem 1.5: Let $(K,<)$ be almost real closed w.r.t some OAG G.
Then $(K,<)$ is st. NIP $\Leftrightarrow$
G is " " .

Therefore the remaining open question is:

Question 1.6: Is every strongly NIP ord. field $(K,<)$ almost r.c. w.r.t. some str. NIP OAG G ?

positive answer to Question 1.6 $\Big\} \Leftrightarrow$ verification of Conj 1.2.

Note: Our theorem 1.5 + AKE + H. A give:

An almost RCF $(K,<)$ is strongly NIP iff $(K,<) \equiv (\mathbb{R}((G)), <)$ where G is one of the gp in H.A.

Proposition 4.9:

Let G strongly NIP and $(K, <)$ almost r.c w.r.t strongly NIP H then

$(K((G)), <)$ is strongly NIP.

Note: $(K((G)), <) \equiv (\mathbb{R}((G \oplus H)), <)$

proof:

H.H. is again str NIP.

$\square$