Making the α-subcompactness of κ indestructible

Bea Adam-Day

University of Leeds

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Indestructibility

We say that an $\mathcal{L}$-large cardinal κ is *indestructible by a class* $\mathcal{A}$ *of forcings* if, after forcing with any $\mathbb{P} \in \mathcal{A}$, κ will remain $\mathcal{L}$-large in the extension.

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Theorem 1.1 (Laver; '79)

After forcing with the Laver preparation $\mathbb{P}_\kappa$, a supercompact cardinal κ will be indestructible under $< \kappa$-directed closed forcing.

Further indestructibility results

Theorem 1.2 (Gitik, Shelah; '89)

One can make the strong compactness of κ indestructible under κ^+-weakly closed forcing satisfying the Prikry Condition.

Theorem 1.3 (Hamkins; '00)

If some amount of GCH is assumed then, using the Lottery Preparation, one can make the λ-supercompactness of κ indestructible by $< \kappa$-directed closed forcing of size at most λ.

Lottery Sums

Definition 1.4

The *lottery sum* of a class $\mathcal{A}$ of forcings is the disjoint sum

$$\oplus\mathcal{A} := \{\langle\mathbb{Q}, p\rangle : \mathbb{Q} \in \mathcal{A} \wedge p \in \mathbb{Q}\} \cup \{\mathbb{1}\}$$

with a new element $\mathbb{1}$ above everything and order given by $\langle\mathbb{Q}, p\rangle \leq \langle\mathbb{R}, q\rangle$ when $\mathbb{Q} = \mathbb{R}$ and $p \leq_{\mathbb{Q}} q$.

Since compatible conditions must have the same $\mathbb{Q}$, the forcing 'holds a lottery' among all forcings in $\mathcal{A}$. The generic filter selects a 'winning' poset and forces with it.

Minimal counterexamples

A *counterexample to the $\mathcal{L}$ largeness of κ* is $(\mathbb{Q}, \lambda, \kappa)$ such that:

1. $\mathbb{Q}$ is a $< \kappa$-directed closed forcing;
2. κ is $\lambda - \mathcal{L}$ large;
3. $\Vdash_{\mathbb{Q}} (\kappa$ is not $\lambda - \mathcal{L}$ large$)$.

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This definition works for large cardinal properties $\mathcal{L}$ where κ being $\lambda - \mathcal{L}$ large implies that κ is $\gamma - \mathcal{L}$ large for all $\gamma < \lambda$.

Supercompact and subcompact cardinals

Definition 2.1 (Magidor Characterisation)

A cardinal κ is λ-supercompact if and only if there exist ordinals $\bar{\kappa} < \bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ with critical point $\bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$.

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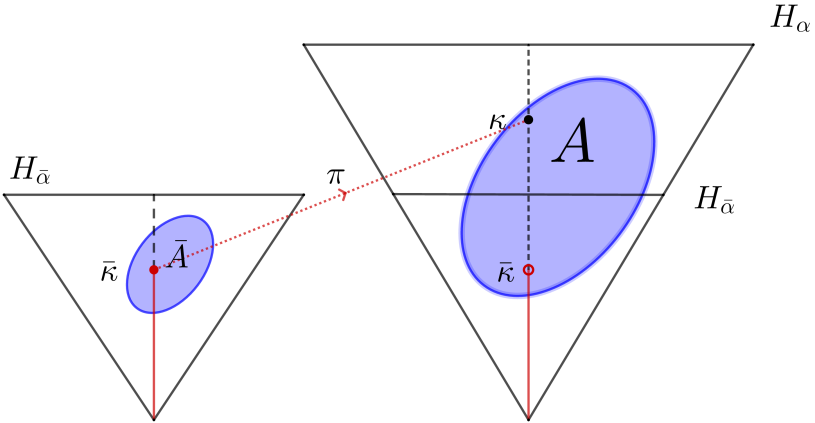
Definition 2.2 (Subcompact Cardinals)

A cardinal κ is α-subcompact for some $\alpha > \kappa$ if for all $A \subseteq H_\alpha$ there exist $\bar{\kappa} < \bar{\alpha} < \kappa$, $\bar{A} \subseteq H_{\bar{\alpha}}$ and an elementary embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)$$

with critical point $\bar{\kappa}$ such that $\pi(\bar{\kappa}) = \kappa$.

Subcompact cardinals



If κ is α-subcompact for some $\alpha > \kappa$ then κ is β-subcompact for all $\kappa < \beta < \alpha$.
If κ is α-subcompact for all $\alpha > \kappa$ then κ is fully supercompact.

The preparatory iteration

Definition 3.1

Fix a cardinal κ and an $\alpha > \kappa$. Define inductively an Easton support iteration $\langle \mathbb{P}^\kappa_\gamma, \dot{\mathbb{Q}}^\kappa_\gamma \rangle_{\gamma<\kappa}$ and a sequence $(\theta^\kappa_\gamma, \eta^\kappa_\gamma)_{\gamma<\kappa}$ as follows: suppose that $\mathbb{P}^\kappa_\delta$ has been defined and that θ^κ_γ, η^κ_γ have been defined for each $\gamma < \delta$.

- If $\delta > \theta^\kappa_\gamma$, η^κ_γ for all $\gamma < \delta$ then let $\dot{\mathbb{Q}}^\kappa_\delta$ denote a $\mathbb{P}^\kappa_\delta$-name for the lottery sum of all forcings $\mathbb{Q}$ with $|\,\mathsf{TC}(\mathbb{Q})| < \kappa$ such that $(\mathbb{Q}, \theta, \delta)$ is a minimal counterexample for some $\theta \leq \kappa$. Let $\eta^\kappa_\delta = |\,\mathsf{TC}(\mathbb{Q})|$ and $\theta^\kappa_\delta = \theta$ for such $\mathbb{Q}$ and θ.
- Otherwise let $\dot{\mathbb{Q}}^\kappa_\delta$ denote a $\mathbb{P}^\kappa_\delta$-name for the trivial forcing and let $\theta^\kappa_\delta = 1 = \eta^\kappa_\delta$.

Some lemmas

Lemma 3.2

$|\mathbb{P}^{\kappa}_{\kappa}| \leq \kappa$. *and we may w.l.o.g. assume that* $\mathbb{P}^{\kappa}_{\kappa} \subseteq H_{\kappa}$. □

We will also need to use the following well-known results.

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Lemma 3.3

If $\mathbb{P}$ *is a forcing notion which doesn't collapse* α *and* $\dot{x} \in H_\alpha$ *then* $\forall p \in \mathbb{P}$, $p \Vdash (\dot{x} \in H_\alpha)$ *i.e.* $\Vdash_{\mathbb{P}} (\dot{x} \in H_\alpha)$. □

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Lemma 3.4

Let α *be a regular cardinal, let* $\mathbb{P} \in H_\alpha$ *be a notion of forcing. Then* $\forall p \in \mathbb{P}$, *if* $p \Vdash (\dot{x} \in H_\alpha)$, *then* $\exists \dot{y} \in H_\alpha$ *such that* $p \Vdash (\dot{x} = \dot{y})$. $\square$

The theorem

Theorem 4.1

Let κ be α-subcompact for some regular cardinal $\alpha > \kappa$. Then, after preparatory forcing with $\mathbb{P}_\kappa^\kappa$, the α-subcompactness of κ will be indestructible under any $< \kappa$-directed closed forcing $\mathbb{Q} \in H_\alpha$.

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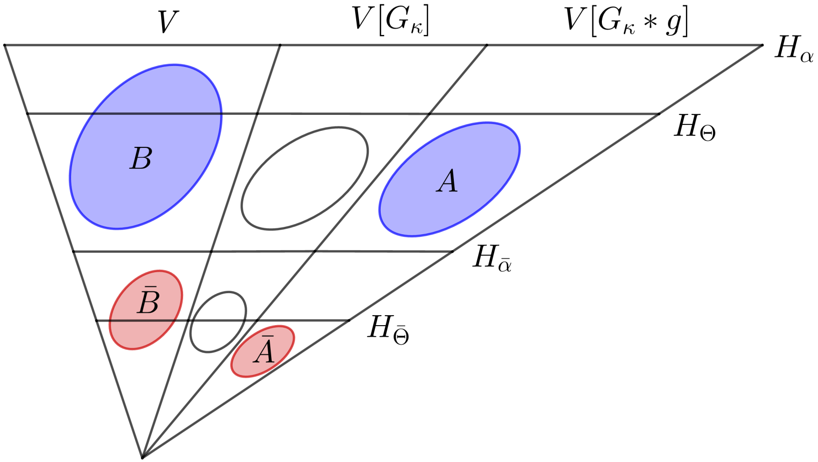
Proof: Suppose not. Then there is a minimal counterexample $(\mathbb{Q}, \Theta, \kappa)$ for some $\Theta \leq \alpha$.

We will show that κ is in fact Θ-subcompact in $V[G_\kappa * g]$, where G_κ is $\mathbb{P}_\kappa^\kappa$-generic over V and g is $\mathbb{Q}$-generic over $V[G_\kappa]$.

Proof sketch



Working in V

So let $A \subseteq H_{\Theta}^{V[G_\kappa * g]}$. Since α is regular and $\mathbb{P}_\kappa^\kappa * \dot{\mathbb{Q}} \in H_\alpha$ we have by Lemma 3.4 that $A = \dot{B}_{G_\kappa * g}$ for some $\dot{B} \subseteq H_\alpha$ in V.

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Since κ is α-subcompact in V, there exist $\bar{\kappa} < \bar{\alpha} < \kappa$, $\bar{B} \subseteq H_{\bar{\alpha}}$ and an α-subcompactness elementary embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{B}) \to (H_\alpha, \in, B)$$

with critical point $\bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$.

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Since κ is α-subcompact in V, there exist $\bar{\kappa} < \bar{\alpha} < \kappa$, $\bar{B} \subseteq H_{\bar{\alpha}}$ and an α-subcompactness elementary embedding

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with critical point $\bar{\kappa}$ and $\pi(\bar{\kappa}) = \kappa$.

Add as a predicate a $\mathbb{P}_\kappa^\kappa$-name, $\mathbb{R}$, that $\mathbb{Q}$ interprets, as well as Θ. So we have

$$\pi : \left(H_{\bar{\alpha}}, \in, \bar{B}, \bar{\mathbb{R}}, \bar{\Theta}\right) \to \left(H_\alpha, \in, B, \mathbb{R}, \Theta\right)$$

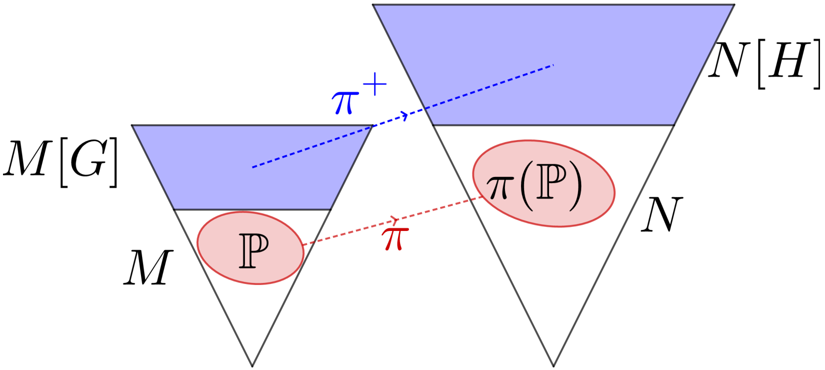
The Lifting Criterion

Theorem 4.2 (The Lifting Criterion)

Let M and N be transitive models of ZFC^-, let $\pi : M \to N$ be an elementary embedding, let $\mathbb{P} \in M$ be a notion of forcing with G generic over $\mathbb{P}$ and let H be $\pi(\mathbb{P})$-generic over N. Then the following are equivalent:

- *there exists an elementary embedding $\pi^+ : M[G] \to N[H]$ with $\pi^+(G) = H$ and $\pi^+ \upharpoonright M = N$*
- *$\pi(p) \in H$ for all $p \in G$*

Lifting diagram



The first lift

Since $\pi(p) = p^\frown \mathbb{1}^{(\kappa)}$ for all $p \in G_{\bar\kappa}$ we may lift the α-subcompactness embedding π in V to

$$\pi^+ : \left(H_{\bar\alpha}[G_{\bar\kappa}], \in, \bar{B}_{G_{\bar\kappa}}, \bar{\mathbb{Q}}, \bar\Theta\right) \to \left(H_\alpha[G_\kappa], \in, B_{G_\kappa}, \mathbb{Q}, \Theta\right)$$

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with critical point $\bar\kappa$ and $\pi^+(\bar\kappa) = \kappa$.

By elementarity $\bar{B} \subseteq H_{\bar\Theta}$ and $(\bar{\mathbb{Q}}, \bar\Theta, \bar\kappa)$ is a minimal counterexample in $V[G_{\bar\kappa}]$ chosen in the stage $\bar\kappa$ lottery by some generic $\bar{g}$.

The second lift

To see that the lifting criterion is again satisfied we use Silver's *master condition* argument – the existence of a master condition for $\bar{\mathbb{Q}}$ and π^+ is guaranteed by the $<\kappa$-directed closure of $\pi^+(\bar{\mathbb{Q}}) = \mathbb{Q}$. So we may lift again to get an α-subcompactness embedding

$$\pi^{++} : \left(H_{\bar{\alpha}}[G_{\bar{\kappa}} * \bar{g}], \in, \bar{B}_{G_{\bar{\kappa}} * \bar{g}}, \bar{\Theta}\right) \to \left(H_{\alpha}[G_{\kappa} * g], \in, B_{G_{\kappa} * g}, \Theta\right)$$

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Let $\bar{A} = \bar{B}_{G_{\bar{\kappa}} * \bar{g}}$ and recall that $B_{G_\kappa * g} = A$ and so π^{++} is in fact an α-subcompactness embedding for A which maps $\bar{A}$ to A, i.e.

$$\pi^{++} : \left(H_{\bar{\alpha}}[G_{\bar{\kappa}} * \bar{g}], \in, \bar{A}, \bar{\Theta}\right) \to \left(H_{\alpha}[G_{\kappa} * g], \in, A, \Theta\right)$$

Restricting the embedding

We have that:

Lemma 4.3

$$H_{\bar{\Theta}}^{V[G_\kappa * g]} = H_{\bar{\Theta}}[G_{\bar{\kappa}} * \bar{g}] \qquad \textit{and} \qquad H_{\Theta}^{V[G_\kappa * g]} = H_{\Theta}[G_\kappa * g]$$

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Using these equalities we may restrict the α-subcompactness embedding embedding in $V[G_\kappa * g]$ to give a Θ-subcompactness embedding

$$\pi^* : \left(H_{\bar{\Theta}}^{V[G_\kappa * g]}, \in, \bar{A}, \bar{\Theta}\right) \to \left(H_{\Theta}^{V[G_\kappa * g]}, \in, A, \Theta\right)$$

with critical point $\bar{\kappa}$ and $\pi^*(\bar{\kappa}) = \kappa$ and so κ is Θ-subcompact in the extension, so a contradiction is reached. □

Thank you for your attention