Geometric triviality in differentially closed fields

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The aim of this talk is to:

- Describe the most important open problem in the study of the theory DCF_0.
- Recall the ω-categoricity conjecture of D. Lascar and the corresponding counterexample of J. Freitag and T. Scanlon
- Describe recent developments around work placing their counterexample into a larger context.
- Point out the main open questions.

1. Refresher on DCF_0.

- The Trichotomy Theorem is arguably the deepest result in the study of DCF_0.

Zilber's Principle holds in DCF_0 (Hrushovski-Sokolovic, 1994)

Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

1. **(Field-like)** *Y is non-orthogonal to the algebraically closed subfield of constants $C_{\mathcal{U}} = \{y \in \mathcal{U} : D(y) = 0\}$; or*
2. **(Group-like)** *Y is non-orthogonal to a very special strongly minimal subgroup of an abelian variety not defined over $C_{\mathcal{U}}$*

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- Except for (3), this gives a full classification of strongly minimal sets in DCF_0.

- We work in the language $L_D = (0, 1, +, \times, D)$ of differential rings.

- DF_0 denotes the theory of differential fields of characteristic zero:

  A **differential field** (K, D) is a field K equipped with a derivation $D : K \to K$, i.e. an additive group homomorphism satisfying the Leibniz rule

$$D(x + y) = D(x) + D(y).$$

$$D(xy) = xD(y) + yD(x).$$

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- This theory can be quite wild: $(\mathbb{Q}, 0, 1, +, \times, D = 0)$ is an example of a differential field and so one gets non-computable definable sets.

- We look at existentially closed models.

- For each $m \in \mathbb{N}_{>0}$, associated with a differential field (K, D), is the ring of differential polynomials in m differential variables,

$$K\{\overline{X}\} = K[\overline{X}, \overline{X}', \ldots, \overline{X}^{(n)}, \ldots].$$

$\bar{X} = (X_1, \ldots, X_m)$ $\bar{X}^{(n)} = (X_1^{(n)}, \ldots, X_m^{(n)})$

$D^n(X) = X^{(n)}$

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- If $f \in K\{\overline{X}\}$ is a differential polynomial, then the order of f, denoted $ord(f)$, is the largest n such that for some i, $X_i^{(n)}$ occurs in f.
- **Example:** $f(X) = (X')^2 - 4X^3 - tX$ is an example of a differential polynomial in $\mathbb{C}(t)\{X\}$ and $ord(f) = 1$.

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- **Example:** $f(X) = (X')^2 - 4X^3 - tX$ is an example of a differential polynomial in $\mathbb{C}(t)\{X\}$ and $ord(f) = 1$.
- The analogue of algebraically closed fields in the differential context is defined as follows

Definition (Blum Axioms, 1968)

A differential field (K, D) is said to be differentially closed if for every $f, g \in K\{X\}$ such that $ord(f) > ord(g)$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

- DCF_0 is the theory obtained by adding to DF_0, the L_D-sentences describing that a differential field is differentially closed.

  It is a very tame theory.

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- In DCF_0 strongly minimal sets determine, in a precise manner, the structure of all definable sets of finite Morley rank.

- In DCF_0, the model theoretic algebraic closure has the following nice characterization:

$$\begin{aligned} acl(A) &= \text{the field theoretic algebraic closure of the differential field generated by } A. \\ &= \mathbb{Q}\left(a, a', a'', \ldots : a \in A\right)^{alg}. \end{aligned}$$

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If Y is a strongly minimal set, then (Y, acl_Y) forms a pregeometry.

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General examples

- ACF_p: $acl =$ field theoretic algebraic closure.
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- The Trichotomy classifies non-trivial strongly minimal sets as essentially the above.

2. Trivial Pursuits

Definition

Let Y be a strongly minimal set. (Y, acl_Y) is (geometrically) trivial if for any $A \subset \mathcal{P}(Y)$ we have that

$$acl_Y(A) = \bigcup_{a \in A} acl_Y(a).$$

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- **Open problem:** Can trivial strongly minimal sets be classified?
  - If there were a strong structure theory, some of the strategy laid out by Hrushovki for certain diophantine problems might be possible.
  - Work on this problem has also lead to proof of important theorems in number theory/functional transcendence in a different direction.

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Old Conjecture (Lascar, 1976)

In DCF_0, geometric triviality $\implies$ ω-categoricity.

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Theorem (Hrushovki, 1995)

The order 1 trivial strongly minimal sets are ω-categorical.

Conjecture (Sadly NOT True)

Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

1. *Y is Field-like; or*
2. *Y is Group-like; or*
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3. *Y has no or little structure, i.e, is ω-categorical.*

- We now know from the work of Freitag and Scanlon that the conjecture is false.
- However, it is still possible that the conjecture is true for order 2 definable sets.

  Indeed, all known examples of order 2 strongly minimal sets in DCF_0 are ω-categorical.

3. Freitag-Scanlon counterexample

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Riemann Mapping Theorem

Let $D \subset \mathbb{C}$ be a simply connected domain that is not $\mathbb{C}$. Then there is a biholomorphic mapping $f : D \rightarrow \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is the complex upper half plane.

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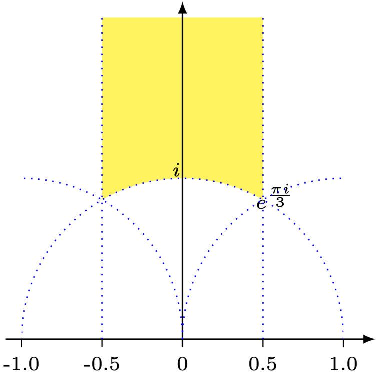
- Recall that we have $SL_2(\mathbb{R}) = \{A \in Mat_2(\mathbb{R}) : det(A) = 1\}$ and its action on $\mathbb{H}$ by linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d} \in \mathbb{H}$$

where $A \in SL_2(\mathbb{R})$ and $\tau \in \mathbb{H}$.

- The Freitag-Scanlon example focuses on the subgroup $SL_2(\mathbb{Z})$. In this case one has that the fundamental domain of action is



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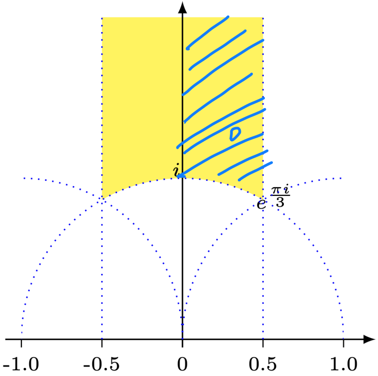

$j: D \longrightarrow H$

- We want to apply the RMT to the fundamental half domain.

$$\text{RMT} \quad \Longrightarrow \quad \exists \quad j : D \to \mathbb{H}$$

$$\downarrow$$

$$\text{RMT} \implies \exists \quad j : D \to \mathbb{H}$$

$\downarrow$ extend

$$\implies \quad j : SL_2(\mathbb{Z}) \setminus \mathbb{H} \to \mathbb{C}$$

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& \Longrightarrow & \quad j : \mathbb{H} \to \mathbb{C} \text{ is } SL_2(\mathbb{Z})\text{-automorphic,} \\
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- The function j is called the modular j-function.

  It satisfies the 3rd order algebraic differential equation

$$\left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 + \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2} \cdot (y')^2 = 0 \qquad (\star_j)$$

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- Other ingredients
  1. Any solution can be taken to be of the form $j(g \cdot t)$ for $g \in SL_2(\mathbb{R})$.
  2. For $N \in \mathbb{N}$, there exist a modular polynomial $\Phi_N \in \mathbb{C}[X, Y]$ such that

$$\Phi_N\,(j(t), j(N \cdot t)) = 0.$$

$N \cdot t = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \cdot t$

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- **Main question:** Is there a way to explain the existence of the modular polynomials?

- $SL_2(\mathbb{Z})$

- 

simply connected
finite

- j-function

General:

- $\Gamma \subset SL_2(\mathbb{R})$ a fuchsian group.

- Γ is of first kind & genus 0

$\subset \mathbb{H}$

- RMT $j_\Gamma : D \to \mathbb{H}$

get $j_\Gamma : \mathbb{H} \to \mathbb{C}$ automorphic for Γ

- j_Γ : uniformizer for Γ

- The function j_Γ satisfies an order 3 ADE of Schwarzian type

$$S_{\frac{d}{dt}}(y) + \frac{1}{2}\sum_{i=1}^{r}\frac{1-{\alpha_i}^2}{(y-a_i)^2} + \sum_{i=1}^{r}\frac{A_i}{y-a_i}\cdot(y')^2 = 0 \qquad (\star_{j_\Gamma})$$

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The definable set X_Γ is strongly minimal and geometrically trivial

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- What about ω-categoricity?

- **Arithmeticity**: An important dividing line in group theory.

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Definition

All subgroups of $SL_2(\mathbb{R})$ obtained this way and their subgroups of finite index are called arithmetic Fuchsian groups.

- Arithmetic Fuchsian groups play an important role in number theory and the quotients $\Gamma \setminus \mathbb{H}$ are known as Shimura Curves of genus 0.

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Theorem (Casale-Freitag-N, 2020)

The set X_Γ is not ω-categorical if and only if Γ is arithmetic.

Arithmetic $\Leftrightarrow$ rich binary structure??

Thm (Poincaré): If j_1 & j_2 are automorphic for Γ

Then j_1 & j_2 are dependent over $\mathbb{C}$

Fact: If $\Gamma_1 < \Gamma$ is of finite index then j_Γ is also automorphic for Γ_1

$\Rightarrow$ IF $\exists\, g \in SL_2(\mathbb{R})$ s.t. $\Gamma_g = \Gamma \cap g\Gamma g^{-1}$ is fin index in both Γ & $g\Gamma g^{-1}$

$\Rightarrow$ the uniformizers $j_\Gamma(t)$ & $j_\Gamma(g^{-1}t)$ are alg dependent over $\mathbb{C}$

$Comm(\Gamma) : \{ g \in SL_2(\mathbb{R}) : \Gamma_g$ fin index in Γ & $g\Gamma g^{-1} \}$

Thm (Margulis): Γ is arithmetic iff Γ has infinite index in $Comm(\Gamma)$

Questions and speculations

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Let Y be a strongly minimal set in $(\mathcal{U}, D) \models DCF_0$. Then exactly one of the following holds:

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1. *Y is Field-like; or*
2. *Y is Group-like; or*
3. *Y arise from an arithmetic Fuchsian group; or*
4. *Y has no or little structure, i.e, is ω-categorical.*

Thank you very much for your attention.