The Split Principle

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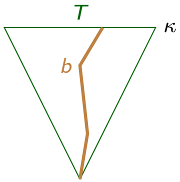
Weak compactness

Definition

A cardinal κ is ***weakly compact*** so long as κ is inaccessible and every κ-tree has a cofinal branch.



Branch property of lists

Let κ be a regular cardinal. We shall refer to a sequence of the form $\langle d_\alpha \mid \alpha < \kappa \rangle$ as a κ-***list*** if for all $\alpha < \kappa$, $d_\alpha \subseteq \alpha$.

Definition

A κ-list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ has a ***cofinal branch*** so long as there is a $b \subseteq \kappa$ such that for all $\gamma < \kappa$ there is an $\alpha \geq \gamma$ such that

$$d_\alpha \cap \gamma = b \cap \gamma.$$

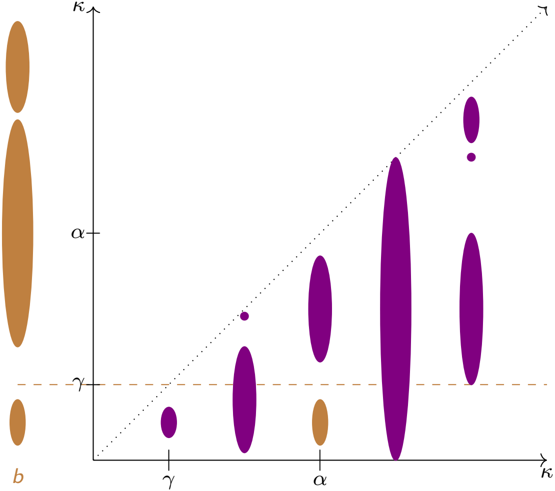
We say that the branch property $\mathrm{BP}(\kappa)$ holds if every κ-list has a cofinal branch.

Given a κ-list $\vec{d}$ we can take characteristic functions $d_\alpha^c : \alpha \to 2$ and look at the binary tree of these characteristic functions:

$$T_{\vec{d}} = \{d_\alpha^c \restriction \beta \mid \beta \leq \alpha < \kappa\} \subseteq {}^{<\kappa}2.$$

Then a function $b : \kappa \to 2$ is a cofinal branch through $T_{\vec{d}}$ so long as for all $\gamma < \kappa$ there is an $\alpha \geq \gamma$ such that $b \restriction \gamma = d_\alpha^c \restriction \gamma$. A κ-list $\vec{d}$ has a cofinal branch iff $T_{\vec{d}}$ does.

If κ is inaccessible, then κ is **weakly compact** $\iff$ $\mathrm{BP}(\kappa)$.


κ
α
γ
γ
α
κ
b

Some other large cardinals

We say that $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ is a $\mathcal{P}_\kappa\lambda$-list if $d_x \subseteq x$ for each $x \in \mathcal{P}_\kappa\lambda$.

If κ is inaccessible, then κ is λ-***compact*** $\iff$ $\mathrm{BP}(\mathcal{P}_\kappa\lambda)$.
A set $B \subseteq \lambda$ is a ***cofinal branch*** through $\mathcal{P}_\kappa\lambda$-list $\vec{d}$ so long as for all $x \in \mathcal{P}_\kappa\lambda$, there is some $y \in \mathcal{P}_\kappa\lambda$ with $y \supseteq x$ such that

$$d_y \cap x = B \cap x.$$

$\mathrm{BP}(\mathcal{P}_\kappa\lambda)$ holds iff every $\mathcal{P}_\kappa\lambda$-list has a cofinal branch.

If κ is inaccessible, then κ is ***ineffable*** $\iff$ $\mathrm{IBP}(\kappa)$.
An ***ineffable branch*** for κ-list $\vec{d}$ is a subset $b \subseteq \kappa$ such that for some stationary $S \subseteq \kappa$ we have that for all $\alpha \in S$,

$$d_\alpha = b \cap \alpha.$$

$\mathrm{IBP}(\kappa)$ holds iff every κ-list has an ineffable branch.

If κ is inaccessible, then κ is λ-***ineffable*** $\iff$ $\mathrm{IBP}(\mathcal{P}_\kappa\lambda)$.
An ***ineffable branch*** for $\mathcal{P}_\kappa\lambda$-list $\vec{d}$ is a subset $B \subseteq \lambda$ such that for some stationary $S \subseteq \mathcal{P}_\kappa\lambda$ we have that for all $x \in S$,

$$d_x = B \cap x.$$

$\mathrm{IBP}(\kappa)$ holds iff every κ-list has an ineffable branch.
If κ is inaccessible, then κ is ***supercompact*** $\iff$ κ is λ-ineffable for unboundedly many λ.

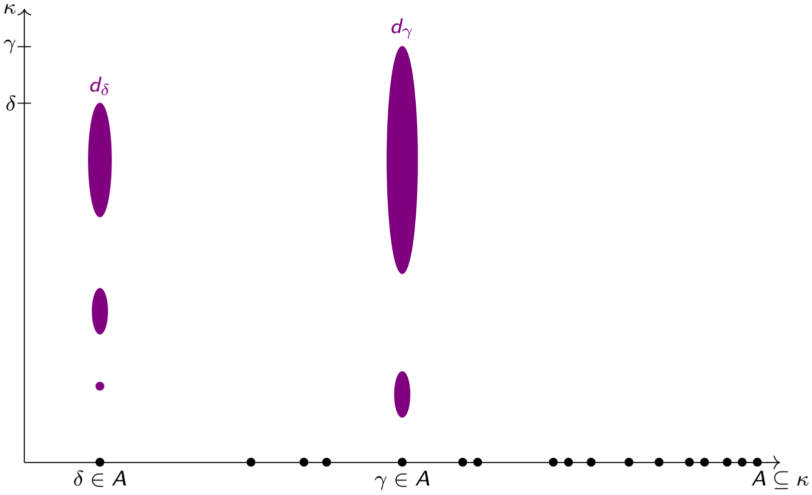
Split(κ)

Definition (The original split principle)

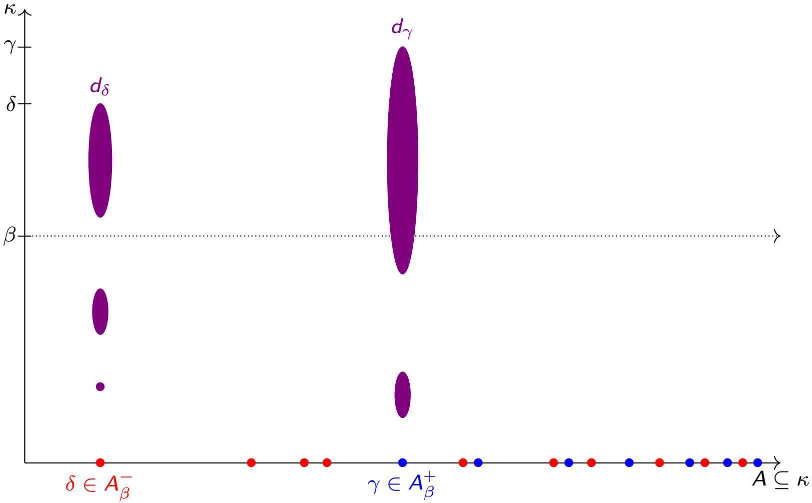
A κ-list $\vec{d}$ ***splits unbounded sets*** so long as for every unbounded $A \subseteq \kappa$ there is a $\beta < \kappa$ which *splits* A; i.e. both

$$A^+_\beta = \{\alpha \in A \mid \beta \in d_\alpha\} \qquad \text{and} \qquad A^-_\beta = \{\alpha \in A \mid \beta \notin d_\alpha\}$$

are unbounded in κ.
If there is a κ-list that splits unbounded sets, then we say that Split(κ) holds.



κ
γ
δ
d_δ
d_γ
$\delta \in A$
$\gamma \in A$
$A \subseteq \kappa$



κ
γ
δ
β
d_γ
d_δ
$\delta \in A_\beta^-$
$\gamma \in A_\beta^+$
$A \subseteq \kappa$

$\mathsf{Split}(\kappa) \iff \neg\,\mathsf{BP}(\kappa)$

For regular κ, the properties of a κ-list splitting unbounded sets and having a cofinal branch are complementary.

Theorem

Let κ be regular, and let $\vec{d}$ be a κ-list. Then

$$\vec{d} \textit{ is a } \mathsf{Split}(\kappa)\textit{-sequence} \iff \vec{d} \textit{ doesn't have a cofinal branch.}$$

Split(κ) $\iff$ $\neg$ BP(κ)

Theorem

Let κ be regular, and let $\vec{d}$ be a κ-list. Then

$$\vec{d} \textit{ is a } \mathrm{Split}(\kappa)\textit{-sequence} \iff \vec{d} \textit{ doesn't have a cofinal branch.}$$

Proof.

($\Longrightarrow$) Assume instead that the Split(κ)-sequence $\vec{d}$ has a cofinal branch, call it b. Define $f : \kappa \to \kappa$ so that

$$f(\gamma) = \text{the least } \alpha \geq \gamma \text{ such that } b \cap \gamma = d_\alpha \cap \gamma.$$

Let $A = f``\kappa$. Since A is unbounded, there is a $\beta < \kappa$ such that

$$A^+_\beta = \{f(\gamma) \in A \mid \beta \in d_{f(\gamma)}\} \qquad \text{and} \qquad A^-_\beta = \{f(\gamma) \in A \mid \beta \in f(\gamma) \setminus d_{f(\gamma)}\}$$

are both unbounded in κ.
Is β in b?

If not: Choose $f(\gamma) \in A^+_\beta$ with $f(\gamma) > f(\beta)$. Then $\gamma > \beta$. But then $\beta \in d_{f(\gamma)} \cap \gamma = b \cap \gamma$, which means $\beta \in b$.

If so: Choose $f(\gamma) \in A^-_\beta$ with $f(\gamma) > f(\beta)$, so $\gamma > \beta$ and $\beta \in b \cap \gamma = d_{f(\gamma)} \cap \gamma$, which means $\beta \in d_{f(\gamma)}$, contradicting $f(\gamma) \in A^-_\beta$.

□

Split(κ) $\iff$ $\neg$ BP(κ)

Theorem

Let κ be regular, and let $\vec{d}$ be a κ-list. Then

$$\vec{d} \text{ is a Split}(\kappa)\text{-sequence} \iff \vec{d} \text{ doesn't have a cofinal branch.}$$

Proof.

($\Longleftarrow$) Assume $\vec{d}$ doesn't split unbounded sets. Let $A \subseteq \kappa$ be unbounded, and no $\beta < \kappa$ splits A; so for each β, exactly one of A^+_β, A^-_β is bounded.
Define a branch $b \subseteq \kappa$ so that

$$\beta \in b \iff A^-_\beta \text{ is bounded } (\iff A^+_\beta \text{ is unbounded}).$$

Note that for each $\beta < \kappa$, there is a least $\xi_\beta < \kappa$ such that either $\beta \in d_\delta$ for all $\delta \in A \setminus \xi_\beta$ or $\beta \notin d_\delta$ for all $\delta \in A \setminus \xi_\beta$.
Let γ be arbitrary. Since κ is regular, there is an $\alpha \in A$ such that $\alpha > \sup_{\beta<\gamma} \xi_\beta$.

Claim: $b \cap \gamma = d_\alpha \cap \gamma$

Pf: Let $\beta < \gamma$. Is $\beta \in b$?

If so: Then A^-_β is bounded, so $\alpha \in A^+_\beta$, and thus $\beta \in d_\alpha$.

If not: Then A^+_β is bounded, so $\alpha \in A^-_\beta$ and thus $\beta \notin d_\alpha$.

□

Generalizations of Split

Definition

Let κ and τ be cardinals, and let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of κ. A κ-sequence $\vec{d}$ of subsets of τ is a $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$-sequence so long as for every $A \in \mathcal{A}$ there is $\beta < \tau$ which splits A into $\mathcal{B}$ with respect to $\vec{d}$, meaning both

$$A^+_\beta = \{\alpha \in A \mid \beta \in d_\alpha\} \qquad \text{and} \qquad A^-_\beta = \{\alpha \in A \mid \beta \notin d_\alpha\}$$

are in $\mathcal{B}$.
$\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$ holds if there is a $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$-sequence.

With this notation, $\text{Split}_\kappa(\textit{unbounded}) \iff \text{Split}(\kappa)$.

Definition

Let κ be regular and $\lambda > \kappa$ be cardinal, and let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of $\mathcal{P}_\kappa\lambda$. A $\mathcal{P}_\kappa\lambda$-sequence $\vec{d}$ is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A},\mathcal{B})$-sequence so long as for every $A \in \mathcal{A}$ there is $\beta < \lambda$ which splits A into $\mathcal{B}$ with respect to $\vec{d}$, meaning both

$$A^+_\beta = \{x \in A \mid \beta \in d_x\} \qquad \text{and} \qquad A^-_\beta = \{x \in A \mid \beta \in x \setminus d_x\}$$

are in $\mathcal{B}$.
$\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A},\mathcal{B})$ holds if there is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A},\mathcal{B})$-sequence.

Large cardinal characterizations

Split principles are "anti-large cardinal axioms" which characterize the failure of a regular cardinal to satisfy certain large cardinal properties.

Theorem

Let κ be a regular cardinal. Then

- κ *is not inaccessible* $\iff$ *for some* $\tau < \kappa$, $\text{Split}_{\kappa,\tau}$(*unbounded*) *holds.*
- κ *is not weakly compact* $\iff$ Split_{κ}(*unbounded*) *holds.*
- κ *is not strongly compact* $\iff$ *for unboundedly many* λ, $\text{Split}_{\mathcal{P}_\kappa\lambda}$(*unbounded*) *holds.*
- κ *is not ineffable* $\iff$ Split_{κ}(*stationary*) *holds.*
- κ *is not* λ*-ineffable* $\iff$ $\text{Split}_{\mathcal{P}_\kappa\lambda}$(*stationary*) *holds.*
- κ *is not supercompact* $\iff$ *for unboundedly many* λ, $\text{Split}_{\mathcal{P}_\kappa\lambda}$(*stationary*) *holds.*

We also found characterizations of subtle cardinals using the ideal derived from split principles, as well λ-Shelah cardinals and some new large cardinals.

Flipping properties (Abramson, Harrington, Kleinberg, Zwicker)

Definition

Let $\vec{X}$ be a sequence of subsets of κ. We say that $\vec{X}'$ is a ***flip*** of $\vec{X}$ if in every coordinate α, either $X'_\alpha = X_\alpha$ or $X'_\alpha = \kappa \setminus X_\alpha$.

- κ is *inaccessible* $\iff$ for every $\alpha < \kappa$ and every α-sequence of subsets of κ there is a flip of the sequence such that the intersection of sets in the flip has cardinality κ.
- κ is *weakly compact* $\iff$ for every κ-sequence of subsets of κ there is a flip of the sequence such that the intersection of any fewer than κ-many sets in the flip has cardinality κ.
- κ is *measurable* $\iff$ given any sequence of subsets of κ, there is a flip such that the intersection of any fewer than κ-many sets in the flip has cardinality κ.

Thank you.