O-minimality of real exponentiation

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1. Preliminaries

Let $\mathcal{L}$ be a language, T be an $\mathcal{L}$-theory.

Definition. T is **model-complete** if for every $M, N \vDash T$, if $M \subseteq N$, then $M \preceq N$.
Fact. T is model-complete if and only if every $\mathcal{L}$-formula is equivalent to a universal $\mathcal{L}$-formula. In particular, if T has quantifier elimination, then it is model complete.

Examples.

- The theory ACF of algebraically closed fields in the language of rings $\mathcal{L}_{\mathrm{ring}} \coloneqq \{0,1,+,\cdot\}$ is model-complete (it has quantifier elimination). Hence the embeddings $\overline{\mathbb{Q}}^{\mathrm{alg}} \subseteq \mathbb{C} \subseteq \overline{\mathbb{C}(t)}^{\mathrm{alg}}$ are all elementary.
- The theory RCF of real closed fields in the language $\mathcal{L}_{\mathrm{ring}}$ is also model-complete. However, it only eliminates quantifiers after moving to $\mathcal{L}_{\mathrm{oring}} \coloneqq \mathcal{L}_{\mathrm{ring}} \cup \{\leq\}$ and adding the axiom $\forall x, y. x \leq y \leftrightarrow \exists z. x + z \cdot z = y$.
  Exercise: verify that (1) RCF does *not* eliminate quantifiers in $\mathcal{L}_{\mathrm{ring}}$, (2) every $\mathcal{L}_{\mathrm{ring}}$-formula is equivalent to an existential formula, and (3) that (2) implies model-completeness.
  Addendum: in the language *of ordered* rings, RCF can be axiomatised by saying that the field is ordered, plus the intermediate value property for polynomials (if $p(a)p(b) < 0$, then there is c between a and b such that $p(c) = 0$). The i.v.p. can be replaced with "every polynomial of odd degree has a zero, and every positive element has a square root". **Exercise:** how would you axiomatise RCF is $\mathcal{L}_{\mathrm{ring}}$ only?

Now assume that $\mathcal{L} \supseteq \{<\}$.
Definition.

- An $\mathcal{L}$-structure M is **o-minimal** if $M \vDash <$ is a total order and every definable subset of M is a finite union of points and intervals. In other words, if every definable subset of M is quantifier-free definable (with parameters) using $<$ only.
- An $\mathcal{L}$-theory is **o-minimal** if every $M \vDash T$ is o-minimal.

- An $\mathcal{L}$-theory is o-minimal if every $M \models T$ is o-minimal.

Remarks.

- Most of the times, o-minimality is also taken to include "$<$ is a dense linear order without endpoints".
- O-minimality is a first order property (Pillay-Steinhorn '88): if M is o-minimal, and $N \equiv M$, then N is o-minimal (in other words, "o-minimality" coincides with "strong o-minimality").
- The type $\operatorname{tp}(a/M)$ of some $a \in N \supseteq M$, where M is o-minimal, is completely determined by the cut of a over M: $\operatorname{cut}_M(a) \coloneqq \{b \in M \mid b < a\}$.

Examples.

- The theory of dense linear orders with or without endpoints. This follows immediately from quantifier elimination.
- $(\omega, <)$, again by quantifier elimination – but no proper expansion is o-minimal (Pillay-Steinhorn '87).
- The theory of real closed fields: it has QE in $\mathcal{L} = \{<, 0, 1, +, \cdot\}$, so every formula is equivalent to a boolean combination of $p(x) > 0$ or $q(x) = 0$, which clearly define finite unions of intervals and points.
- What we are going to see today.

1. Restricted analytic functions…

Definitions. Let

- $\mathbb{R}\{X_1, \dots, X_n\}$, for $n \geq 0$, be the ring of functions $[-1,1]^n \to \mathbb{R}$ that are *analytic* on some open $U \supseteq [-1,1]^n$ (where U may depend on the function).
- $\mathcal{L}_{\mathrm{an}} \coloneqq \mathcal{L}_{\mathrm{oring}} \cup \{\tilde{f}\}_{f \in \mathbb{R}\{X_1, \dots, X_n\}, n \in \mathbb{N}}$, where $\tilde{f}$ are function symbols with the obvious arities.
- $\mathbb{R}_{\mathrm{an}}$ be the structure obtained by interpreting $\mathcal{L}_{\mathrm{oring}}$ as usual and each $\tilde{f}$ as the function f.
  Examples: we add symbols $\widetilde{\cos}$, $\widetilde{\sin}$, $\widetilde{\exp}$ for the functions $\cos_{\restriction[-1,1]}$, $\sin_{\restriction[-1,1]}$, $\exp_{\restriction[-1,1]}$, as well as for every constant function.
- T_{an} be the complete $\mathcal{L}_{\mathrm{an}}$-theory of $\mathbb{R}_{\mathrm{an}}$.

Theorems.

- Gabrielov '68: T_{an} is model-complete and o-minimal (noted by van den Dries '86).
- Denef-van den Dries '88: T_{an} eliminates quantifiers after adding a binary $D(x, y) = \frac{x}{y}$ for $|x| \leq |y| \leq 1$ and $y \neq 0$, $D(x, y) = 0$ otherwise. Moreover, they describe a natural complete axiomatisation.

Corollaries. T_{an} cannot define:

- Global $\cos$ and $\sin$, otherwise it would not be o-minimal (the set $\cos(x) = 0$ is not a finite union of points and intervals).
- Global $\exp$, because T_{an} is also **polynomially bounded**: every definable unary function is eventually dominated by a polynomial (i.e., for every definable f there is some n such that $|f(x)| \leq x^n$ for $x \to +\infty$).
- In fact, every definable function $\mathbb{R} \to \mathbb{R}$ coincides with an $\mathcal{L}_{\text{an}}^{D}$-term (to be defined later) for $x \gg 1$.

2. ...and real exponentiation

Definitions.

- $\mathcal{L}_{\exp} \coloneqq \mathcal{L}_{\text{oring}} \cup \{\exp\}$, where $\exp$ is a unary function symbol.
- $\mathbb{R}_{\exp}$ be the structure obtained by interpreting $\exp$ as real exponentiation.
- $T_{\exp}$ be the complete $\mathcal{L}_{\exp}$-theory of $\mathbb{R}_{\exp}$.
- $\mathcal{L}_{\text{an,exp}} \coloneqq \mathcal{L}_{\text{an}} \cup \mathcal{L}_{\exp}$.
- $\mathbb{R}_{\text{an,exp}}$ be the common expansion of $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\exp}$.
- $T_{\text{an,exp}}$ be the complete $\mathcal{L}_{\text{an,exp}}$-theory of $\mathbb{R}_{\text{an,exp}}$.

Theorems.

- Wilkie '94 (but more like '91-92): $T_{\exp}$ is model-complete and o-minimal.
- Van den Dries-Miller '94: $T_{\text{an,exp}}$ is model-complete and o-minimal.
- Van den Dries-Macintyre-Marker '94 (after Ressayre '93): $T_{\text{an,exp}}$ eliminate quantifiers after adding a unary $\log$. Moreover, they describe a universal axiomatisation in that language.

3. The axiomatisation

Recall that $\mathbb{R}\{X_1, \dots, X_n\}$ is a ring. But you can also *compose* functions, provided the image of the inner function falls within $[-1,1]^n$.

More precisely, let's keep in mind that for any $f \in \mathbb{R}\{X_1, \dots, X_n\}$, we can compose f in at least the following two ways:

(i) Let $g_1, \dots, g_n \in \mathbb{R}[X_1, \dots, X_m]$ be such that $g_i([-1,1]^m) \subseteq [-1,1]$ and $g_i(0) = 0$ for all i. Then $f \circ (g_1, \dots, g_n)_{\restriction[-1,1]^m} \in \mathbb{R}\{X_1, \dots, X_m\}$.

(ii) Let $\bar{a} \in [-1,1]^n$ and $\varepsilon \in \mathbb{R}^{>0}$ such that $\bar{a} + \varepsilon[-1,1]^n \subseteq [-1,1]^n$. Then $f(\bar{a} + \varepsilon\bar{X}) \in \mathbb{R}\{X_1, \dots, X_n\}$.

Theorem (van den Dries-Macintyre-Marker '94, plus Ressayre '93). $T_{\text{an,exp}}$ is axiomatised by the following schemes.

(a) The axioms of ordered fields.

(b) Each positive element has an n-th root for all $n \geq 2$ [actually redundant here – see below].
(c) (AC1-2) The map sending f to the interpretation of $\tilde{f}$ is a ring homomorphism mapping X_i to the function x_i, and (AC3-4) it preserves the partial compositions as in (i)-(ii).
(d) (E1-3) The map exp is an ordered group isomorphism from the *additive group* to the positive part of the *multiplicative group*, i.e. $\exp(x+y) = \exp(x)\exp(y)$, exp is injective and surjective over the positive elements.
(e) (E4) $x > n^2 \to \exp(x) > x^n$ for all $n \in \mathbb{N}$ and all x.
(f) (E5) $-1 \leq x \leq 1 \to \exp(x) = \widetilde{\exp}(x)$.

Moreover, the above axiomatisation is *universal* after adding $\log$ to the language.

Addendum. Note that (E1-3) and (E5) already determine exp completely on $\mathbb{R}$: for every $r \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $\frac{r}{n} \in [-1,1]$, hence $\exp(r) = \widetilde{\exp}\left(\frac{r}{n}\right)^n$. However, you need more info when you go to a non-standard model. One can construct explicit models of (E1-3)+(E5) where, for instance, $\exp(x) = x$ has cofinally many solutions.

Remark. (b) is redundant because of $x^{\frac{1}{n}} = \exp\left(\frac{1}{n}\log(x)\right)$. I report it here for completeness: (a)-(c) is an axiomasition of T_{an}.

We shall now walk through the key steps in van den Dries-Macintyre-Marker '94 towards the proof of the above theorem.

We shall use, without proof, that the axiomatisation (a)-(c) of T_{an} is o-minimal, model-complete, and has QE + universal axiomatisation after adding the definable function D to the language, where $D(x,y) = \frac{x}{y}$ for $|x| \leq |y| \leq 1$ and $y \neq 0$, $D(x,y) = 0$ otherwise (Denef-van den Dries '88).

4. The Archimedean valuation

Definitions.

- Let K be a field, G be an ordered group. A **valuation** is a map $v: K^\times \to G$ such that
  a. $v(xy) = v(x) + v(y)$
  b. $v(x+y) \geq \min\{v(x), v(y)\}$ (ultrametric inequality).

  One may also define $v(0) = \infty = +\infty$ to patch up a value at 0.
  Exercise: check that the balls $B_a(g) \coloneqq \{x \in K \mid v(x-a) > g\}$ form a basis for a topology on K under which $+$ and $\cdot$ are continuous. Observe that two balls can only be disjoint or contained in one another.

- Suppose K is ordered. For $x, y \in K^{\wedge}$, let $x \preceq y$ if $|x| \leq n|y|$ for some $n \in \mathbb{N}$, and $x \asymp y$ if $x \preceq y \preceq x$. The quotient $(K^\times/\asymp, \succeq)$ is an ordered group (note the flipped order) and the map $K \to K^\times/\asymp$ is called Archimedean valuation. **Exercise:** verify that it is a valuation.
- From now on, denote by v the Archimedean valuation.

Addendum. The field $\mathbb{R}$ has trivial Archimedean valuation: the quotient $\mathbb{R}^\times/\asymp$ consists of a single point. The ordered field $\mathbb{R}(t)$, where by convention $0 < t < r$ for all $r \in \mathcal{R}^{>0}$, has Archimedean value group $\mathbb{Z}$: each $\asymp$-equivalence class is represented by t^n for some $n \in \mathbb{Z}$, and if you let $v(t^n) = n$; note for instance that $v(t^n t^m) = n + m$. **Exercise:** verify explicitly that for every $f \in \mathbb{R}(t)$ there is a unique $n_f \in \mathbb{Z}$ such that $f \asymp t^{n_f}$; define $v(f) = n_f$ and verify that the map $v: \mathbb{R}(t) \to \mathbb{Z}$ is a valuation.

Remark. A valuation is measuring the size of an element: $v(x)$ is very large when x is very small, as in close to zero. Hence, $v(x - y)$ is very large when x is close to y.

Lemma 3.4. Let $K \subseteq F$ be an extension of real closed fields and $y \in F \setminus K$. If $v(K(y)^\times) \neq v(K^\times)$, then there is $a \in K$ such that $v(y - a) \notin v(K^\times)$.
Proof. Let $\frac{p(y)}{q(y)} \in K(y)^\times$ be an element with valuation outside of $v(K^\times)$. Since $v\left(\frac{p(y)}{q(y)}\right) = v\big(p(y)\big) - v\big(q(y)\big)$, we may assume that $v\big(p(y)\big) \notin v(K^\times)$ for some polynomial $p(Y) \in K[Y]$. Since K is real closed, we may assume that $p(Y)$ is either $(Y - a)$ or $(Y - a)^2 + b^2$. In the former case, we are done. In the latter, if by contradiction $v(y - a) \in v(K^\times)$, then $v\big((y - a)^2\big) = v(b^2)$, hence $v\big(p(y)\big) > v(b^2)$, but $0 < b^2 < p(y)$, a contradiction. ■

Now suppose $M, N \vDash T_{\text{an}}$. For $y \in N \setminus M$, denote by $M\langle y\rangle$ the *definable closure* of $M \cup \{y\}$ into N. Note that $M\langle y\rangle$ is automatically an $\mathcal{L}_{\text{an}}$-substructure and a subfield.

Lemma 3.7. Let $M, N \vDash T_{\text{an}}$ with $M \subseteq N$ and $y \in N \setminus M$. Then $v(M\langle y\rangle^\times)$ is the divisible hull of $v(M(y)^\times)$.
Proof sketch. $v(M\langle y\rangle^\times)$ obviously contains $v(M(y)^\times)$, and it is divisible, because n-th roots of positive elements are definable.
For the other inclusion: each unary definable function f can be expanded as a *Puiseux* series (think Taylor series but with fractional exponents). Hence $f(y) \sim ay^q$ for some $a \in M$, $q \in \mathbb{Q}$, by which $v\big(f(y)\big) = v(a) + qv(y)$*. For a better argument: if $G = v(M^\times) \oplus \mathbb{Q}v(y)$, then there is an $\mathcal{L}_{\text{an}}$-embedding of $M\langle y\rangle^\times$ into $\mathbb{R}\big((t^G)\big)_{\text{an}}$ (see Lemma 3.3). ■

*This is also saying that T_{an} is "power-bounded" with field of powers $\mathbb{Q}$. **Exercise:** Let M be an o-minimal expansion of a field. Consider the definable ~~auto~~endomorphisms of the ordered group $(M^{>0},\cdot,<)$. Show that it has a natural field structure (what are sum and product?), called **field of powers**. (Hint: prove that the endomorphisms embed into M. How? An endomorphism is a function "like" $x \mapsto x^{\alpha}$. Can you recover α?) Verify, based on Lemma 3.7, that T_{an} has indeed field powers $\mathbb{Q}$ and it is **power-bounded**: every definable function is eventually dominated by a power.

5. Quantifier elimination

Q.E. is based on the following observation. Call $\mathcal{L}_{\mathrm{an,log}} \coloneqq \mathcal{L}_{\mathrm{an}} \cup \{\log\}$, $\mathcal{L}_{\mathrm{an,exp,log}} \coloneqq \mathcal{L}_{\mathrm{an,exp}} \cup \mathcal{L}_{\mathrm{an,log}}$.

Theorem 4.1. Let $K \vDash T_{\mathrm{an,exp}}$, F_0 be an $\mathcal{L}_{\mathrm{an,log}}$-substructure of K with $F_0 \vDash T_{\mathrm{an}}$. If L is a $|K|^+$-saturated model of $T_{\mathrm{an,exp}}$ and $\sigma_0 : F_0 \to L$ is an $\mathcal{L}_{\mathrm{an,log}}$-embedding, then σ_0 can be extended to an $\mathcal{L}_{\mathrm{an,log}}$-embedding of K into L.

First, why does it imply quantifier elimination?

Corollary 4.5. $T_{\mathrm{an,exp}}$ admits quantifier elimination in $\mathcal{L}_{\mathrm{an,exp,log}}$.
Proof. Take models M, N with N $|M|^+$-saturated. Take an embedding $\sigma : A \to N$ of some substructure $A \subseteq M$. By the axiomatisation of T_{an}, $A \vDash T_{\mathrm{an}}$. By 4.1, we may extend σ to an embedding of N. This implies QE: the truth of existential formulas with parameters in A is determined by the isomorphism type of A! **Exercise:** fill out the (purely model theoretic) details.

To prove Theorem 4.1, one proceeds *one element at a time*. In the following, take K, F_0, L, σ_0 as in 4.1.

Lemma	Assumption	Extend σ_0 to
4.2	$v(F_0(y)^{\times}) = v(F_0^{\times})$	$F = F_0\langle y\rangle$
	By 3.7, every $w \in F$ can be written as $w = z(1+\varepsilon)$ with $z \in F_0$ and $\varepsilon \prec 1$. Thus $\log\big(z(1+\varepsilon)\big) = \log(z) + \widetilde{\log}(1+\varepsilon)$ (using (E1-3,5)). Therefore, F is an $\mathcal{L}_{\mathrm{an,log}}$-structure. σ_0 extends to an $\mathcal{L}_{\mathrm{an}}$-embedding $F \to L$ by model-completeness of T_{an}. By the above formula, the extension is also an $\mathcal{L}_{\mathrm{an,log}}$-embedding.	
4.3	$v(F_0(x)^{\times}) \neq v(F_0^{\times})$ for all $x \in K \setminus F_0$	$F = F_0\langle \exp(y)\rangle$

4.3	$v(F_0(x)^\times) \neq v(F_0^\times)$ for all $x \in K \setminus F_0$ $y \in F_0$ with $\exp(y) \notin F_0$	$F = F_0\langle\exp(y)\rangle$

By 3.7, $v(F) = v(F_0^\times) \oplus \mathbb{Q}g$ where $g = v(\exp(y))$. Thus, every $w \in F_0\langle\exp(y)\rangle$ can be written as $w = z(1+\varepsilon)\exp(qy)$ with $z, \varepsilon \in K$, $q \in \mathbb{Q}$, hence F is an $\mathcal{L}_{\text{an,log}}$-structure. Now map $\exp(y)$ to $\exp(\sigma_0(y))$. One can easily verify that they realise the same *cut* over F_0. By o-minimality and model-completeness of T_{an}, we get an $\mathcal{L}_{\text{an}}$-embedding, which happens to be an $\mathcal{L}_{\text{an,log}}$-embedding as well by the above formula.

4.4	As 4.3, plus F_0 closed under exp, $y \in K \setminus F_0$.	Some $\mathcal{L}_{\text{an,log}}$-structure $F_0(y) \subseteq F \subseteq K$.

We may assume $v(y) < 0$. One can build a sequence as follows:

- Let $y_0 := y, y_1 := \log(y_0)$.
- Assume we have y_n. By 3.4, let $\beta_n \in F_0$ such that $v(\log(y_n) - \beta_n) \notin v(F_0^\times)$. Let $y_{n+1} := |\log y_n - \beta_n|$, so that $\log y_n = \beta_n + \varepsilon_n y_{n+1}$.
- Let $F = F_0\langle y_0, y_1, \dots\rangle$.
- By (E4), we have $v(y_0) < v(y_1) < v(y_2) < \cdots < 0$. Moreover, the values are $\mathbb{Q}$-linearly independent over $v(F_0^\times)$. By 3.7, we have $v(F^\times) = v(F_0) \oplus \mathbb{Q}v(y_0) \oplus \mathbb{Q}v(y_1) \oplus \cdots$
- Pick a realisation a of the cut of y over F_0 in L. One verifies that each $a_1 := \log(a)$, $a_{n+1} := \varepsilon_n(\log(a_n) - \sigma_0(\beta_n))$ verifies the same cut as y_n over F_0, hence by o-minimality and model-completeness of T_{an} one can extend σ_0 to an $\mathcal{L}_{\text{an}}$-embedding of F.
- Every $w \in F$ can we written as $w = z(1+\varepsilon)y_0^{q_0}y_1^{q_1}\cdots$. Hence σ_0 is also an $\mathcal{L}_{\text{an,log}}$-embedding.

6. Hardy fields and o-minimality

Definitions.

- Let $\mathcal{R}$ be some expansion of the ordered field $(\mathbb{R}, <, 0, 1, +, \cdot, \dots)$ **with no additional relation symbols**, let T be the complete theory of $\mathcal{R}$.
- Let $\mathcal{G}$ be the ring of germs of functions $f : \mathbb{R} \to \mathbb{R}$. A germ or equivalence

- Let $\mathcal{G}$ be the ring of germs of functions $f, g: \mathbb{R} \to \mathbb{R}$. A **germ** an equivalence class for the relation $f \sim g$ when $f(x) = g(x)$ for all $x \gg 1$.
- An **$\mathcal{R}$-field** is a subfield of $\mathcal{G}$ that is closed under (the germs of) all the functions in the language of $\mathcal{R}$ (of any arity).
- A **Hardy field** is a subfield of $\mathcal{G}$ that is closed under differentiation.
- Given $K \subseteq \mathcal{G}$ and $g \in \mathcal{G}$, we say that g is **comparable** to K if for all $f \in K$, either ultimately $g(x) < f(x)$, or ultimately $g(x) > f(x)$, or ultimately $g(x) = f(x)$.

Fact/exercise. A subfield of $\mathcal{G}$ must have the following property: for every f in the subfield, either $f(x) > 0$, $f(x) = 0$, or $f(x) < 0$ for all $x \gg 1$. In other words, every element must be comparable to $\{0\}$.

Lemma 5.2. If T has quantifier elimination, then T is o-minimal if and only if each *term* in one variable is eventually positive, negative, or zero (i.e. comparable to $\{0\}$).
Proof. Exercise!

Lemma 5.5. If T has quantifier elimination and there exists an $\mathcal{R}$-field containing $\mathbb{R}(x)$, then T is o-minimal.
Proof. Since it is a field, every element is comparable to $\{0\}$. Since it is an $\mathcal{R}$-field containing $\mathbb{R}(x)$, it contains the germs of all terms in one variable. By 5.2, T is o-minimal. ■

Now, let us assume that T has QE, as well as a universal axiomatisation (so that substructures are automatically models, hence elementary substructures).
Exercise: prove that substructures are indeed elementary substructures under the above assumptions. Show an example of a theory with QE where some substructures are not always elementary (and thus, the theory does not have a universal axiomatisation).

Lemma 5.8. If K is an $\mathcal{R}$-field, then K can be naturally viewed as a model of T.
Proof. Since K is closed by all functions in the language, it is naturally a structure in the language of $\mathcal{R}$. Suppose $T \vdash \forall \bar{x} \bigvee_{i=1}^{M} \bigwedge_{j=1}^{N} \varphi_{ij}(\bar{x})$ (with φ_{ij} atomic or negated atomic) and take $\bar{f} \in K^{|\bar{x}|}$. Consider the (definable) function $x \mapsto i(N+1)+j$, picking the least $i(N+1)+j$ such that $\mathcal{R} \vDash \varphi_{ij}\left(\bar{f}(x)\right)$. By QE, this function eventually coincides with a term. Since K is an $\mathcal{R}$-field, and the range is finite, it is eventually equal to some $i(N+1)+j$, in which case $K \vDash \varphi_{ij}(\bar{f})$. Therefore, $K \vDash \forall \bar{x} \bigvee_i \bigwedge_j \varphi_{ij}(\bar{x})$. Since T has a universal axiomatisation, $K \vDash T$. [Note: DMM uses a different argument.] ■

Lemma 5.9. Let T be o-minimal and let K be an $\mathcal{R}$-field. If $g \in \mathcal{G}$ is comparable with K, then the "$\mathcal{R}$-field generated by g over K", denoted by $K\langle g\rangle$, exists.
Proof. Let $K\langle g\rangle$ be the closure of $K \cup \{g\}$ under all terms. Since g is comparable with K, it determines a cut over K. By o-minimality, the composition of all terms with g is eventually positive, negative, or zero. Then $K\langle g\rangle$ is a field, hence it is an $\mathcal{R}$-field. ■

Lemma 5.11. Let K be a Hardy field and $f \in K$.

1. $e^{f(x)}$ is comparable with K.
2. If $f > 0$, then $\log(f(x))$ is comparable with K.

Proof. 1. Suppose not. Then for some $g \in K$, $e^f - g = e^f(1 - e^{-f}g)$ keeps changing sign as $x \to \infty$. Hence the same holds for $1 - e^{-f}g$, as well as for its derivative $e^{-f}(f'g - g')$. But then $f'g - g'$ keeps changing sign as $x \to \infty$, a contradiction since K is a Hardy field.
2. First, we verify that given $f, g \in K$, the function $\left(\int f\right) - g$ eventually stops changing sign. Suppose not: then its derivative $f - g'$ keeps changing sign, against the assumption that K is a Hardy field. Since $\log(f) = \int f'/f$, this shows that $\log(f)$ is comparable with K. ■

Lemma 5.12. Let T be o-minimal and let K be an $\mathcal{R}$-field containing. Pick $f \in K$. Then $K\langle e^f\rangle$ is an $\mathcal{R}$-Hardy field, and if $f > 0$, likewise for $K\langle\log(f)\rangle$.
Proof. By o-minimality, K is also closed under derivations, hence it is an $\mathcal{R}$-Hardy field. Thus, by 5.11, $K\langle e^f\rangle$ is an $\mathcal{R}$-field. Moreover, again by 5.11, e^f determines a cut over K. This is enough to show that every element of $K\langle e^f\rangle$, which can be expressed as a term in $K \cup \{f\}$, can be differentiated yielding another element of $K\langle e^f\rangle$. Therefore, $K\langle e^f\rangle$ is an $\mathcal{R}$-Hardy field. ■

Corollary 5.13. $\mathbb{R}_{\text{an,exp}}$ is o-minimal.
Proof. We know that T_{an} is o-minimal, admits QE and a universal axiomatisation in the appropriate language. Then it has an $\mathcal{R}$-field containing $\mathbb{R}(x)$: just take $\mathbb{R}\langle x\rangle$, per 5.9. Now apply 5.12 repeatedly until we obtain an $\mathcal{R}$-field closed under exp and log. Since $T_{\text{an,exp}}$ admits QE, it is o-minimal by 5.5. ■