Building countable generic structures

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Fraïssé construction

- $\mathfrak{L}$ relational language.
- $\mathcal{C}$ a class of finite $\mathfrak{L}$-structures is a Fraïssé class if it is closed under *substructure* and *isomorphism* and:
    1. If $A, B \in \mathcal{C}$ then there is $C \in \mathcal{C}$ such that $A, B \subseteq C$ (JEP);
    2. If $A, B, C \in \mathcal{C}$ and $A \subseteq B, C$, then there is $D \in \mathcal{C}$ such that $B \subseteq D$ and $C \subseteq D$ (AP).
- We assume $\emptyset \in \mathcal{C}$ (JEP is implied by AP).

Theorem

(Fraïssé 1953) Suppose $\mathcal{C}$ is a Fraïssé class. Then, there exists a unique, up to isomorphism, countable structure $\mathfrak{A}$ that is ultra-homogeneous and $\mathcal{C} = \mathrm{Age}(\mathfrak{A})$.

- A structure is ultra-homogeneous if every isomorphism between finite substructures extends to an *automorphism*.
- $\mathfrak{A}$ is called the Fraïssé limit of $\mathcal{C}$, denoted by $\mathit{Flim}(\mathcal{C})$.

Fraïssé limits

Examples

- Class of all finite linearly ordered sets; Fraïssé limit is $(\mathbb{Q}, <)$.
- Class of all finite graphs; Fraïssé limit is called the Random graph.
- Class of all finite metric spaces with rational distances; Fraïssé limit is Rational Urysohn space which the completion is the Urysohn universal space (metric space that contains all separable metric spaces).
- ...

- A countable structure $\mathfrak{A}$ with $\mathrm{Age}(\mathfrak{A}) = \mathcal{C}$ is ultra-homogenous if and only if $A \subseteq \mathfrak{A}$ and $A \subseteq B \in \mathcal{C}$, then there is an embedding $f : B \longrightarrow \mathfrak{A}$ with $f \restriction_A = \mathrm{id}_A$ and $f[B] \subseteq \mathfrak{A}$ (*richness*).

Richness

- Let $\mathfrak{A}$ be a structure with $\mathcal{C} \subseteq \mathrm{Age}(\mathfrak{A})$.
- $\mathfrak{A}$ is $\mathcal{C}$-rich if $A \subseteq \mathfrak{A}$ and $A \subseteq B \in \mathcal{C}$, then there is an embedding $f : B \longrightarrow \mathfrak{A}$ with $f \restriction_A = \mathrm{id}_A$ and $f[B] \subseteq \mathfrak{A}$.

Smooth class

- $\mathfrak{L}$ relational language.
- $\mathcal{K}$ a class of finite $\mathfrak{L}$-structures that is closed under isomorphism and substructure.
- Let $\leqslant$ be a *reflexive* and *transitive* relation on elements of $A \subseteq B$ of $\mathcal{K}$ and moreover, invariant under $\mathfrak{L}$-embeddings.
- The class $(\mathcal{K}, \leqslant)$ is called a smooth class if
  1. $\emptyset \in \mathcal{K}$; and $\emptyset \leqslant A$ for all $A \in \mathcal{K}$;
  2. If $A \subseteq B' \subseteq B$, then $A \leqslant B$ implies that $A \leqslant B'$;
  3. If $A, A_1, A_2 \in \mathcal{K}$ and $A_1, A_2 \subseteq A$, then $A_1 \leqslant A$ implies $A_1 \cap A_2 \leqslant A_1$.
- $A \in \mathcal{K}$ and $\mathfrak{N}$ be an $\mathfrak{L}$-structure with $\mathrm{Age}(\mathfrak{N}) \subseteq \mathcal{K}$. Define $A \leqslant \mathfrak{N}$ if and only if $A \leqslant B$ for every finite $B \subseteq \mathfrak{N}$ such that $A \subseteq B$.

Generic structures

- $(\mathcal{K}, \leqslant)$ has AP if for every $A, B, C \in \mathcal{K}$ with $A \leqslant B, C$, there is $D \in \mathcal{K}$ such that $B, C \leqslant D$.

Theorem

Suppose $(\mathcal{K}, \leqslant)$ is a smooth class with AP, then there is a unique countable structure $\mathfrak{M}$, up to isomorphism, satisfying:

1. $\mathrm{Age}(\mathfrak{M}) = \mathcal{K}$;
2. *If $A \leqslant \mathfrak{M}$ and $A \leqslant B \in \mathcal{K}$, then there is an embedding $f : B \longrightarrow \mathfrak{M}$ with $f \restriction_A = id_A$ and $f[B] \leqslant \mathfrak{M}$;*
3. $\mathfrak{M} = \bigcup_{i \in \omega} A_i$ *where $\langle A_i; i \in \omega \rangle$ is $\leqslant$-chain of finite subsets of $\mathfrak{M}$.*

- We call the structure $\mathfrak{M}$ obtained from the theorem above $(\mathcal{K}, \leqslant)$-generic structure.
- Every Fraïssé class is a smooth class with AP.

Pre-dimension functions

- Let K be the class of all finite graphs and $\alpha \in (0,1)$.
- Define $\delta_\alpha : \mathrm{K} \longrightarrow \mathbb{R}$ as $\delta_\alpha(A) = |A| - \alpha \cdot |E(A)|$ where $E(A)$ is the set edges of A.
- For every $A \subseteq B \in \mathrm{K}$, define $A \leqslant_\alpha B$ if and only if $\delta_\alpha(C) - \delta_\alpha(A) \geq 0$ for every C with $A \subseteq C \subseteq B$.
- $(\mathrm{K}, \leqslant_\alpha)$ not a smooth class (JEP also fails).
- Let $\mathrm{K}_\alpha := \{A \in \mathrm{K} : \delta_\alpha(A') \geq 0, \text{ for every } A' \subseteq A\}$.
- The class $(\mathrm{K}_\alpha, \leqslant_\alpha)$ is a smooth class with AP. Let M^α for the countable $(\mathrm{K}_\alpha, \leqslant_\alpha)$-generic structure.
- (Baldwin-Shi) $Th(\mathrm{M}^\alpha)$ is stable of infinite Morley rank; ω-stable when α is rational.
- (Baldwin-Shelah) When α is irrational $Th(\mathrm{M}^\alpha)$ is the almost sure theory of first order sentences of random graphs with edge probability $n^{-\alpha}$.

New method

- $\mathfrak{L}$ relational language and $\mathcal{K}$ a class of finite $\mathfrak{L}$-structures which is closed under isomorphism and substructure.
- Let $\mathcal{I} \subseteq \mathcal{K} \times \mathcal{K}$ be such that if $(A, B) \in \mathcal{I}$, then $A \subseteq B$ and $A \neq \emptyset$. Let $B^* := B \backslash A$.
  1. If $(A, B) \in \mathcal{I}$ and $B = A_0 B^* \otimes_{A_0} C$ where $A_0 \subseteq A$ then $(A_0, A_0 B^*) \in \mathcal{I}$;
  2. If $(A, B) \in \mathcal{I}$ and $A \subseteq C$, then $(C, C \otimes_A B) \in \mathcal{I}$;
  3. If $(A, B) \in \mathcal{I}$ and $B = B_1 \otimes_{A_1} B_2$ where $A_1 \supseteq A$ then $(A_1, B_i) \in \mathcal{I}$ for $i = 1, 2$.
- For $C, D \in \mathcal{K}$ with $C \subseteq D$ define

$$C \leqslant^{\mathcal{I}} D \text{ iff } \forall (A, B) \in \mathcal{I} \text{ with } B \subseteq D \text{ if } A \subseteq C, \text{ then } B \subseteq C.$$

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- Every Fraïssé class $\mathcal{C}$ is a $\left(\mathcal{C}, \leqslant^{\emptyset}\right)$ a smooth class with AP.

New method

Theorem

Suppose $\mathcal{K}$ is the class of all finite $\mathfrak{L}$-structures. Then $(\mathcal{K}, \leqslant^{\mathcal{I}})$ is a smooth class with AP.

- $(\mathcal{K}, \leqslant^{\mathcal{I}})$-generic is not necessarily saturated.
- One way to overcome this problem is to restrict to $\mathcal{K}^{\mu} \subset \mathcal{K}$ where $\mu : \omega \times \omega \to \omega$ and for every $D \in \mathcal{K}^{\mu}$ and every $(A, B) \in \mathcal{I}$ with $B \subseteq D$ the supremum of the cardinals of maximal of families disjoint copies of B over A in D is bounded by $\mu(|A|, |B|)$. This property is often called the algebraic closure property.
- Generic structures of classes with the algebraic closure property and full-amalgamation property are saturated.

Example

- K be the class of all finite graphs and $\alpha \in (0,1)$.
- $\delta_\alpha : \mathrm{K} \longrightarrow \mathbb{R}$ as $\delta_\alpha(A) = |A| - \alpha \cdot |E(A)|$.
- $\mathrm{K}_\alpha := \{A \in \mathrm{K} : \delta_\alpha(A') \geq 0, \text{ for every } A' \subseteq A\}$.
- $A, B \in \mathrm{K}_\alpha$ where $A \subseteq B$. We say (A, B) is a co-minimal pair if
  1. $\delta_\alpha(B/A) < 0$;
  2. $\delta_\alpha(B'/A) \geq 0$ for all $A \subseteq B' \subset B$.
- Let $\mathcal{I}$ be the collection of all co-minimal pairs (A, B) where $A, B \in \mathrm{K}_\alpha$.
- $A, B \in \mathrm{K}_\alpha$ then $A \leqslant_\alpha B$ iff $A \leqslant^{\mathcal{I}} B$.
- $(\mathrm{K}, \leqslant^{\mathcal{I}})$ is a smooth class with AP.
- $(\mathrm{K}, \leqslant^{\mathcal{I}})$-generic structure has IP.
- $(\mathrm{K}_{\alpha'}, \leqslant^{\mathcal{I}})$ is a smooth class with AP for every $0 \leq \alpha' \leq \alpha$.

Structural Ramsey theory

- A topological group G is a group with a topology on G such that the group's operation and the inverse function are continuous.
- A continuous action of G on a compact Hausdorff space is called a G-flow.
- G is called extremely amenable if every G-flow has a fix point.
- G is amenable if every G-flow supports a G-invariant Borel probability measure.

Extreme-amenability and amenability

Theorem

Suppose $G = \mathrm{Aut}(\mathfrak{A})$*, where* $\mathfrak{A} = \mathit{Flim}(\mathcal{C})$

- *(Kechris, Pestov, Todorcevic '05) G is extremely amenable if and only if G preserves a linear order on* $\mathfrak{A}$ *and the ordered* $\mathcal{C}$ *has Ramsey property.*
- *(Moore '10) G is amenable if and only* $\mathcal{C}$ *has the convex Ramsey property.*

Theorem

(Gh., Khalilian, Pourmahdian '15) Suppose $G := \mathrm{Aut}(\mathfrak{M})$ *where* $\mathfrak{M}$ *is* $(\mathcal{K}, \leqslant)$*-generic structure. Then*

- *G is extremely amenable if and only if G preserves a linear order on* $\mathfrak{M}$ *and the ordered* $(\mathcal{K}, \leqslant)$ *has Ramsey property.*
- *G is amenable if and only if the class* $(\mathcal{K}, \leqslant)$ *has the convex Ramsey property.*

Ramsey property

- Suppose $(\mathcal{K}, \leqslant)$ is a smooth class.
- Let $A, B \in \mathcal{K}$ and $A \leqslant B$. Let $\binom{B}{A}$ be the set of all $\leqslant$-embeddings of A in B.
- Write $C \to (B)^A_2$ when for every $f : \binom{C}{A} \to \{0, 1\}$ there is $B' \in \binom{C}{B}$ such that $\left\{f(A') : A' \in \binom{B'}{A}\right\}$ is monochromatic
- Suppose $(\mathcal{K}, \leqslant)$ is a Ramsey class if for every $A, B \in \mathcal{K}$ with $A \leqslant B$ there is $C \in \mathcal{K}$ such that $C \to (B)^A_2$.

Examples

- Class of all finite linearly ordered sets.
- Class of all finite ordered graphs.
- Class of all finite ordered metric spaces with rational distances.

- (Prömel) $C \nrightarrow (B)^a_2$ when $m(C) < \frac{1}{2} r\eta^*(B)$ for some $r \geq 2$.

- $m(C)$ is the *maximum density* $:= \max\{\frac{|E(B)|}{|B|} : B \subseteq C\}$.
- $\eta(A) = \min\{deg(a) : a \in A\}$ and $\eta^*(C) := \max\{\eta(A) : A \subseteq C\}$.
- (Prömel) $C \nrightarrow (B)^a_2$ when $m(C) < \frac{1}{2} r\eta^*(B)$ for some $r \geq 2$.

Corollary

(Gh., Khalilian, Pourmahdian '15) The automorphism groups of Hrushovski generic structures that are obtained from pre-dimension functions are not extremely-amenable.

Proof.

Take L be a loop of n vertices ($n \geq 3$). Singletons are $\leqslant_\alpha$-closed in L. $\nu^*(L) = 2$. Note that $m(C) \leq \alpha$ for every $C \in \mathrm{K}_\alpha$. Choose $r > \alpha$ and then the Ramsey property fails for $(\mathrm{K}_\alpha, \leqslant_\alpha)$. □

Theorem

(Gh., Khalilian, Pourmahdian '15- Evans, Hubička, Nešetřil '17)
The automorphism groups of Hrushovski generic structures that are obtained from pre-dimension functions with rational coefficients for both cases of are not amenable.

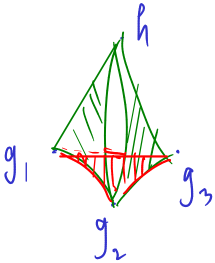
- There is a pair $(A; B)$ where $A, B \in \mathrm{K}_\alpha$ forms a free 2-pseudo-plane in M^α.

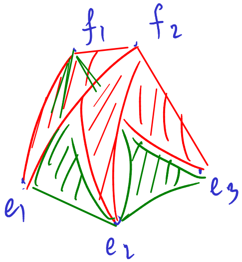
Corollary

The automorphism group of $(\mathrm{K}, \leqslant^{\mathcal{I}})$*-generic structure is not amenable.*

More examples

- $\mathfrak{L} := \{\mathcal{R}\}$ where $\mathcal{R}$ is ternary, irreflexive and symmetric.
- Let $\mathcal{K}$ be the class of all finite $\mathfrak{L}$-structures.
- Let $\mathcal{I} := \{(G, H), (E, F)\}$ where $G \subseteq H$ and $E \subseteq F$ with the following diagrams
  1. $H = \{g_1, g_2, g_3, h\}$ and $G = \{g_1, g_2, g_3\}\}$ with $\mathcal{R}(H) = \{(g_1, g_2, g_3), (h, g_1, g_2), (h, g_3, g_2)\}$.
  2. $F = \{e_1, e_2, e_3, f_1, f_2\}$ and $E = \{e_1, e_2, e_3\}$ with $\mathcal{R}(F) = \{(e_1, e_2, f_1), (e_2, e_3, f_1), (e_1, f_1, f_2), (e_2, f_1, f_2), (e_3, f_1, f_2)\}$.





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- For $A, B \in \mathcal{K}$ with $A \subseteq B$, define $A \leqslant^{\mathcal{I}} B$.
- $(\mathcal{K}, \leqslant^{\mathcal{I}})$ is a smooth class with AP.
- There is an infinite increasing chain of co-minimal pairs.
- The $(\mathcal{K}, \leqslant^{\mathcal{I}})$-generic structure is not saturated.

- Let $\chi^{\mathcal{D}}(A, B) :=$ max cardinality of $B^1, \ldots, B^n$ subsets of D with $B^i \cong_A B$ where $B^i \cap B^j = A$ for $i \neq j$.
- $\mathcal{K}^\eta := \{A \in \mathcal{K} : \ \forall \ (X, Y) \in \mathcal{I} \text{ with } Y \subseteq A, \ \chi^A(X, Y) \leq 2\}$.
- $(\mathcal{K}^\eta, \leqslant^{\mathcal{I}})$ is a smooth class with AP.
- $(\mathcal{K}^\eta, \leqslant^{\mathcal{I}})$-generic structure is not simple and has TP_2.
- $(\mathcal{K}^\eta, \leqslant^{\mathcal{I}})$-generic structure has $NSOP_1$.
- $A \mathop{\smile\!\!\!\!\!\mid}_C^{f-a} B$ if and only if $acl(AC) \cap acl(BC) = acl(C)$ and $acl(AC)$ is free from $acl(BC)$ over $acl(C)$.

Further discussions

- Saturated generic structure - full amalgamation property.
- Collapsed (identify the zero-extension) and ω-categorical.
- Generalised measurability in the sense of Anscombe-Macpherson-Steinhorn-Wolf.
- Simplicity of the automorphism groups
- Group topology - they are ND-classes i.e. Zariski topology is not group topology.
- Algebraic closure - closure operator with *finite character*
- Uncountable case and AEC's - Tarsi-Vaught chain axiom - quasi excellence. classes with finitary cl.

Thank you very much for your attention!