Double-Membership Graphs of Models of Anti-Foundation

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“*Some questions about sets are irrelevant to mathematics. [One such question is] is there an x such that $x = \{x\}$?*”

K. Kunen

Motivation

Definition 1

The symmetrised membership graph of a countable model of ZFC has vertices corresponding to sets and an edge between vertices x and y if $x \in y$ or $y \in x$.

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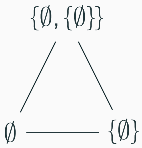
Figure 1: The symmetrised membership graph of the ordinals up to 2.



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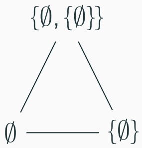

Recall

The Random Graph is the graph obtained with probability 1 on a countable vertex set, where pairs of distinct vertices are adjoined with probability 1/2.

Motivation

A property of graphs is the *Alice's Restaurant Property* (ARP) which characterises (up to isomorphism) the Random Graph:

> *Given any two disjoint finite sets U and V of vertices, there is a new vertex z adjacent to all vertices in U but none in V.*

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The symmetrised membership graph of any countable model of ZFC *is isomorphic to the Random Graph*

This is true since the ARP holds in the membership graph. The proof doesn't use much set theory, but it does rely in a crucial way upon the **Axiom of Foundation**.

Definitions

An alternative to Foundation is the Anti-Foundation Axiom:

Definition 3

Let X be a set of 'indeterminates', and A a set of sets. A *flat system of equations* is a set of equations of the form $x = S_x$, where S_x is a subset of $X \cup A$ for each $x \in X$.

A *solution* f is a function taking elements of X to sets such that, after replacing each $x \in X$ with $f(x)$, all its equations become true.

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AFA

The *Anti-Foundation Axiom* (**AFA**) is the statement that every flat system of equations has a unique solution.

ZFA is Zermelo-Fraenkel set theory with Foundation replaced by Anti-Foundation, and is equiconsistent with **ZFC**.

Consider the flat system with $X = \{x, y\}$, $A = \{\emptyset, \{\emptyset\}\}$ and the equations $x = \{x, y, \emptyset\}$ and $y = \{x, \{\emptyset\}\}$.

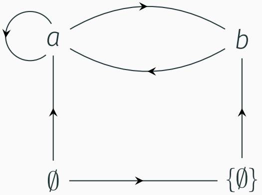

Figure 2: A picture of sets a and b satisfying the flat system.

Membership Graphs

Let $L = \{\in\}$, where $\in$ is a binary relation symbol, and M an L-structure. Let S and D be the definable relations

$$S(x, y) := x \in y \vee y \in x$$
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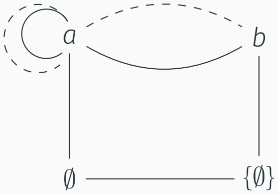
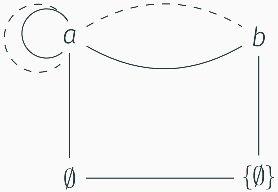


The SD-graph of *a* and *b*.



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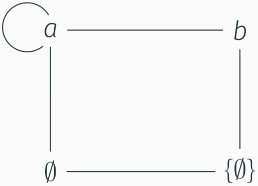
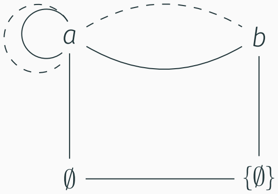


The S-graph of *a* and *b*.

Examples



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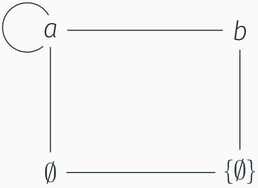

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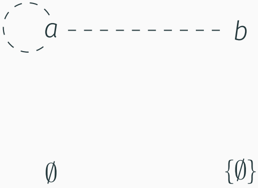

The D-graph of a and b.

Single Membership Graphs

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Like the Random Graph, the Random Loopy Graph is $\aleph_0$-categorical, ultrahomogeneous and supersimple.

The analogous *Loopy Alice's Restaurant Property* is:

> *For any two finite disjoint sets U and V there are vertices z_1 and z_2, where z_1 has a loop and z_2 does not, such that both are adjacent to all vertices of U but none of V.*

It characterises (up to isomorphism) the Random Loopy Graph.

S-Graphs

Let M be a countable model of ZFA.

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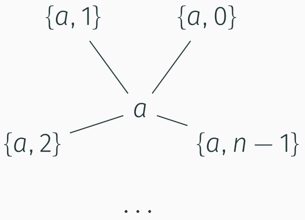
However, neither the SD-Graph nor the D-Graph of a countable model of **ZFA** is $\aleph_0$-categorical, since for each n there is a distinct 1-type in the graph given by $a = \{\{a, i\} \mid i < n\}$:



Double Membership: Number of Graphs

Connected Components

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The theorem is generalised and used to characterise the connected components of M_D as follows.

Theorem 8 (A-D, Howe, Mennuni)

Up to isomorphism, the connected components of M_D are exactly the connected graphs in the sense of M. In particular there are infinitely many copies of each of them.

Let M be a model of ZFA.

Theorem 9 (A-D, Howe, Mennuni)

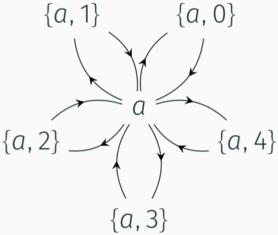
There are $2^{\aleph_0}$ pairwise non-isomorphic countable models of ZFA whose D-graphs (resp. SD-graphs) are elementarily equivalent to M_D (resp. M_{SD}).

Let M be a model of **ZFA**.

Theorem 9 (A-D, Howe, Mennuni)

*There are $2^{\aleph_0}$ pairwise non-isomorphic countable models of **ZFA** whose D-graphs (resp. SD-graphs) are elementarily equivalent to M_D (resp. M_{SD}).*

Proof sketch: Let $n \in \omega \setminus \{0\}$. An *n-flower* is a point with degree n in the D-graph M_D of M but no loop. E.g. $a = \{\{a, i\} \mid i < 5\}$ is a 5-flower:



Proof sketch continued

For A a subset of $\omega \setminus \{0\}$, a point b is an *A-bouquet* if it has no loop, has D-edges to at least one n-flower for each $n \in A$, but not to any n-flower for $n \notin A$.

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Define $\beta_A(y)$ to be the set of L_D-formulae such that $b \in M_D$ is an A-bouquet iff for all $\psi(y) \in \beta_A(y)$ we have $M_D \models \psi(b)$.

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In Th(M_D) the $2^{\aleph_0}$ sets of formulae β_A are each consistent – by compactness it suffices to show that if $A_0 \subseteq A$ is finite and $A_1 \subseteq \omega \setminus (\{0\} \cup A)$ is finite then there is a point $b \in M$ with **a D-edge to an n-flower for all $n \in A_0$ and no D-edges to n-flowers for $n \in A_1$.**

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An A_0-bouquet satisfies this requirement.

For distinct $A, B \subseteq \omega \setminus \{0\}$ we may wlog assume $\exists n \in A \setminus B$. Then $\beta_A(y) \vdash$ "$\exists$ an n-flower joined to y", which contradicts $\beta_B(y) \vdash$ "there is no n-flower joined to y". So the β_A are pairwise contradictory.

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Thus $\mathrm{Th}(M)$ has $2^{\aleph_0}$ distinct countable models, as each can only realise countably many of the β_A.

The reducts to L_{SD} (or L_D) of models realising different β_A are still non-isomorphic as the β_A are partial types in the language L_D. $\square$

Double Membership: Common Theory

Incompleteness

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Proof sketch: Code consistency statements in the D-Graph and use the existence of a consistency statement independent of **ZFA**. □

Further, each of the completions of $\mathrm{Th}(K_D)$ are the theory of some D-graph M_D and, in fact:

Theorem 11 (A-D, Howe, Mennuni)

*For $M, N \models$ **ZFA**, $M_D \equiv N_D$ if and only if M and N satisfy the same consistency statements.*

So the only obstruction to completeness is that models may not satisfy the same consistency statements.

Non-elementarity

Recall that there is only one countable S-graph of $M \models \mathbf{ZFA}$, up to isomorphism.

Theorem 12 (A-D, Howe, Mennuni)

*For every $M \models$ **ZFA** there exist*

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Proof sketch:

- A D-graph of a model of **ZFA** necessarily contains a connected component of infinite diameter.
- Adapt the proof of Hanf's Theorem to show the existence of $N \equiv M_{SD}$ where $N \restriction_{L_D}$ has no connected component of infinite diameter. □

Open Problems

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Thank you for your attention!

Peter Aczel.
Non-Wellfounded Sets, volume 14 of *CSLI Lecture Notes*.
CSLI Publications, 1988.

Bea Adam-Day and Peter Cameron.
Undirecting membership in models of ZFA.
https://arxiv.org/abs/1708.07943, preprint, 2017.

Bea Adam-Day, John Howe, and Rosario Mennuni.
On double-membership graphs of models of anti-foundation.
https://arxiv.org/abs/1908.02708, preprint, 2019.

Jon Barwise and Lawrence S. Moss.
Vicious Circles, volume 60 of *CSLI Lecture Notes*.
CSLI Publications, 1996.

Peter Cameron.
The Random Graph.
In Roland L. Graham, Jaroslav Nešetřil, and Steve Butler, editors, *The Mathematics of Paul Erdős*, volume 2, pages 353–378. Springer, 2013.

Marco Forti and Furio Honsell.
Set theory with free construction principles.
Annali della Scuola Normale Superiore di Pisa — Classe di Scienze, 10:493–522, 1983.