Undirecting Membership in Models of Anti-Foundation

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Background Information

- **The Axiom of Foundation**

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$$\forall x(x \neq \emptyset \to \exists y \in x(y \cap x = \emptyset))$$

- $\in$**-graphs**/containment graphs of sets.
- **The Radom Graph R** is a graph with countably many vertices and an edge between any two vertices with probability $\frac{1}{2}$.

Alice's Restaurant Property

The Alice's Restaurant Property:
A graph G has the ARP if, given finitely many distinct vertices $\{x_0, \ldots, x_{n-1}\}$ and $\{y_0, \ldots, y_{m-1}\}$ there exists a new vertex z which is adjacent to all the x_i and nonadjacent to all the y_j.
In such cases we say that z is 'correctly joined'.

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Lemma

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Lemma

With probability 1, a countable random graph has the ARP.

Lemma

Any two countable graphs with the ARP are isomorphic.

Theorem (Cameron)

Let M be a countable model of ZFC. The $\in$-graph of M is isomorphic to the Random Graph.

Properties of the Random Graph R

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- R is $\aleph_0$**-categorical**
  i.e any countable model that satisfies the same first-order sentences as R is isomorphic to R.
- R is $\aleph_0$**-universal**
  i.e any finite or countable graph may be embedded as an induced subgraph of R.

The Anti Foundation Axiom (AFA)

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Definition

Let X be a set of 'indeterminates', let A be a set of sets called 'atoms'.

A flat system of equations is a set of equations of the form $x = S_x$ where $S_x \subseteq X \cup A$ for each $x \in X$.

A solution to a flat system of equations is an assignment of sets to the indeterminates so that the equations become true.

The Loopy Alice's Restaurant Property

Definition

A graph X has the Loopy Alice's Restaurant Property if for all distinct subsets $\{x_0, \ldots, x_{n-1}\}$ and $\{y_0, \ldots, y_{m-1}\}$ there exist new vertices z_1 and z_2 which are both adjacent to all the x_i and none of the y_j, and where z_2 has a loop but z_1 does not.

- With probability 1, a countable random graph with loops has the Loopy ARP
- Any two countable graphs with the loopy ARP are isomorphic

Replacing Foundation with Anti Foundation

Theorem (A-D, Cameron)

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Lemma

In a model of a subset of ZF including at least Specification and Union, there is no set S which contains all sets of cardinality p, for fixed $p \in \mathbb{N}$.

Properties or the Random Loopy Graph

- The Random Loopy Graph is $\aleph_0$-homogeneous
- The Random Loopy Graph is $\aleph_0$-categorical
- The Random Loopy Graph is $\aleph_0$-universal

The Double-Edged Case

Theorem (A-D, Cameron)

The membership graph of a countable model of ZFA, keeping double edges, is not $\aleph_0$-categorical.

Theorem (Engeler, Ryll-Nardzewski, Svenonious)

For a countable first order structure M, it is $\aleph_0$-categorical if and only is it has only finitely many n-types for every n.

Open Questions

- Is it true that, if two countable multigraphs are elementarily equivalent, and one is the membership graph of a model of ZFA, then so is the other?
- Is it true that there are infinitely many non-isomorphic graphs which are membership graphs of countable models of ZFA?