Indestructibility of Supercompactness

Bea Adam-Day

mmbiad@leeds.ac.uk

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Supercompact Cardinals

Definition

An ultrafilter U over $\mathcal{P}_\kappa(\gamma)$ is *fine* if it's κ-complete (i.e. every sequence of length $< \kappa$ has a lower bound in U) and also for all $a \in \gamma$:

$$\{x \in \mathcal{P}_\kappa(\gamma) \ : \ a \in x\} \in U$$

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Definition

A fine ultrafilter U over $\mathcal{P}_\kappa(\gamma)$ is *normal* if it's closed under diagonal intersection, i.e. for all $\langle X_i : i < \gamma \rangle \in^\gamma U$,

$$\Delta_{i<\gamma} X_i := \{x \in \mathcal{P}(\gamma) : x \in \cap_{i \in x} X_i\} \in U$$

Definition

A cardinal κ is γ-*supercompact* for some cardinal $\gamma \geq \kappa$ if there exists a normal ultrafilter over $\mathcal{P}_\kappa(\gamma)$.

A cardinal κ is *supercompact* if κ is γ-supercompact for all $\gamma \geq \kappa$.

The Ultrapower Embedding

For U a (normal) ultrafilter on $\mathcal{P}_\kappa(\gamma)$ the canonical ultrapower embedding is given by:

$$\begin{aligned} j_U : V &\to M_U \cong \mathrm{Ult}(V,U) \\ x &\mapsto [c_x]_U \end{aligned}$$

where c_x is the constant function $c_x : \mathcal{P}_\kappa\gamma \to \{x\}$

Two-stage Iterated Forcing

Definition

Let $\mathbb{P}$ be a notion of forcing in M, a transitive model of ZFC.

- A $\mathbb{P}$-name $\dot{x}$ is *canonical* if there does not exist $\dot{y}$ such that $|\operatorname{TC}(\dot{y})| < |\operatorname{TC}(\dot{x})|$ and $\Vdash_{\mathbb{P}} (\dot{y} = \dot{x})$.
- If $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a notion of forcing then $\mathbb{P} * \dot{\mathbb{Q}}$ is the set of pairs $(p, \dot{q})$ where $p \in \mathbb{P}$, $\Vdash_{\mathbb{P}} (\dot{q} \in \dot{\mathbb{Q}})$ and $\dot{q}$ is canonical.
- The order is given by: $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2)$ if and only if $p_1 \leq p_2$ and $p_1 \Vdash (\dot{q}_1 \leq \dot{q}_2)$.

n-stage Iterated Forcing

Similarly for $n \in \omega$ we define $\mathbb{P}_n = \mathbb{P}_{n-1} * \dot{\mathbb{Q}}$ where $\mathbb{P}_{n-1}$ is a notion of forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}_{n-1}$-name for a notion of forcing. The elements of $\mathbb{P}_n$ are n-tuples $\rho = \langle \rho_0, \rho_1, \cdots, \rho_{n-1} \rangle$[1]

[1] Technically $p = \langle\langle \cdots \langle\langle \rho_0, \dot{\rho_1} \rangle, \dot{\rho_2} \rangle \cdots \dot{\rho_{n-2}} \rangle \dot{\rho_{n-1}} \rangle$

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For p a condition in $\mathbb{P}_\alpha$ define its support to be $\mathrm{supp}(p) = \{\beta < \alpha \colon p(\beta) \neq \dot{1}_{\mathbb{Q}_\beta}\}$.

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But what happens at limit stages?

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Direct and Inverse Limits

Definition

For λ a limit ordinal and an iteration of length λ

- the *inverse limit* is

$$\varprojlim \mathbb{P} \restriction_\lambda = \{\text{sequences } p \text{ of length } \lambda \text{ such that } p \restriction_\alpha \in \mathbb{P}_\alpha \forall \alpha < \lambda\}$$

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- the *direct limit* is

$$\varinjlim \mathbb{P} \restriction_\lambda = \left\{p \in \varprojlim \mathbb{P} \restriction_\lambda \text{ s.t. } p(\alpha) = \dot{1}_{\mathbb{Q}_\alpha} \forall \alpha \text{ sufficiently large}\right\}$$

Choosing the right limits

It turns out that taking lots of direct limits (stationarily many) preserves the κ-chain condition (κ-c.c.) in the forcing extension. Similarly taking lots of inverse limits (at all limit stages of cofinality $< \kappa$) preserves κ-closure.

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We will be using an iteration with *Easton support*; an iteration where direct limits are taken at regular limit stages and inverse limits are taken elsewhere.

This choice gives the right balance between chain condition and closure requirements, and so is good for arguments involving large cardinal preservation.

Laver's Theorem

Theorem [R. Laver, 1978]

For κ supercompact, there exists a κ-c.c. partial ordering $\mathbb{P}$ with $|\mathbb{P}| = \kappa$ such that, in $V^{\mathbb{P}}$, κ is supercompact and remains supercompact upon forcing with any κ-directed closed partial ordering.

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Here a poset $\mathbb{P}$ is κ-directed closed if every directed set D of size $< \kappa$ has a lower bound in $\mathbb{P}$ i.e. $\exists p \in \mathbb{P}$ such that $p \leq d$ for all $d \in D$.

A Key Lemma

In order to prove Laver's result we first have to prove the following Lemma:

Lemma

Let κ be supercompact. Then there exists an $f : \kappa \to V_\kappa$ such that for every set x and every $\lambda \geq |\mathrm{TC}(x)|$, there is a normal ultrafilter U_λ on $\mathcal{P}_\kappa(\lambda)$ with

$$(j_\lambda f)(\kappa) = x$$

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Such a function is known as a *Laver function*.

Proof of Lemma

We argue by contradiction: suppose the Lemma does not hold.

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We now define $\Phi(g, \delta_0)$ to be the statement:

$$\exists\alpha \text{ such that } g : \alpha \to V_\alpha \wedge$$
$$\delta_0 = \min\Big\{\delta \in \mathrm{On} : \exists x \text{ with } \delta \geq |\operatorname{TC}(x)| \wedge \nexists U_\delta$$
$$\text{over } \mathcal{P}_\alpha(\delta) \text{ s.t. } (j_\delta g)(\alpha) = x\Big\}$$

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Thus with this notation, we are assuming for contradiction that $\Phi(f, \lambda_f)$ holds for all $f : \kappa \to V_\kappa$.

Now let ν be greater than all the λ_f and let U_ν be a normal ultrafilter on $\mathcal{P}_\kappa(\nu)$.

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Now let U_κ be the projection of U_ν on κ i.e.

$$U_\kappa = \{X \subseteq \mathcal{P}_\kappa(\kappa) \colon \pi^{-1}(X) \in U_\nu\}$$

(Here the projection $\pi : \mathcal{P}_\kappa\gamma \to \mathcal{P}_\kappa\kappa$ takes x to $x \cap \kappa$.)

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(Here the projection $\pi : \mathcal{P}_\kappa\gamma \to \mathcal{P}_\kappa\kappa$ takes x to $x \cap \kappa$.)

Further let $A \in U_\kappa$ be the set of α where for each $g : \alpha \to V_\alpha$ there is some $\lambda_g < \kappa$ such that $\Phi(g, \lambda_g)$.

$A \in U_\kappa$

Note that this A is in U_κ since the normality of the ultrafilter U_ν gives that

$$X \in U_\nu \iff j_\nu{}''(\nu) \in j_\nu(X)$$

where $j_\nu{}''(\nu) = \{j_\nu(\alpha) : \alpha < \nu\}$ is the pointwise image of ν under j_ν. We will use this characterisation of normality throughout.

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By definition of the projection, if $\pi^{-1}(A) \in U_\nu$ then $A \in U_\kappa$. Thus to show that $A \in U_\kappa$ it suffices to show that $j_\nu''(\nu) \in j_\nu(\pi^{-1}A) = \{x \in \mathcal{P}_{j_\nu(\kappa)} j_\nu(\nu) : x \cap j_\nu(\kappa) \in j_\nu(A)\}$

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Note that the critical point of j_ν is κ and so $j_\nu{}''(\nu) \cap j_\nu(\kappa) = \kappa$. So it suffices to show that $\kappa \in j_\nu(A)$. Since $\Phi(f, \lambda_f)$ holds for all $f : \kappa \to V_\kappa$ and since $j_\nu(\kappa) > \kappa$, the result follows.

Defining the Laver Function

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- If not then let $f(\alpha) = \varnothing$.

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- If not then let $f(\alpha) = \varnothing$.

Let U_{λ_f} be the projection of U_ν onto λ_f. We claim that

$$(j_{\lambda_f} f)(\kappa) = x$$

which will contradict that x is a witness to $\Phi(f, \lambda_f)$ and so we will be done.

Deriving a Contradiction

To prove the claim first note that, in the canonical commutative diagram, the function $h : M_{\lambda_f} \to M_\nu$ is the identity on λ_f.

Further, ${}^{\lambda_f}M_{\lambda_f} \subseteq M_{\lambda_f}$, $|\,\mathrm{TC}(x)| \leq \lambda_f$ and $x \in M_{\lambda_f}$ and so we have that $h(x) = x$.

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This gives that:

$$(j_{\lambda_f} f)(\kappa) = (h^{-1} j_{\lambda_f})(\kappa) = h^{-1}(j_\nu f(\kappa)) = h^{-1}(x) = x$$

which proves the claim and gives a contradiction as required. $\square$

Proof of Theorem

Now we may prove the theorem itself, namely:

For κ supercompact, there exists a κ-c.c. partial ordering $\mathbb{P}$ with $|\mathbb{P}| = \kappa$ such that, in $V^{\mathbb{P}}$, κ is supercompact and remains supercompact upon forcing with any κ-directed closed partial ordering.

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First we let $f : \kappa \to V_\kappa$ be a Laver function. We will define an Easton extension of length κ, which will be the *Laver preparation*. For each $\alpha < \kappa$ we will affiliate with $\mathbb{P}_\alpha$ an ordinal λ_α.

The Laver Preparation

- For limit γ take $\mathbb{P}_\gamma$ to be sequences in the inverse limit of $\langle \mathbb{P}_\beta \colon \beta < \gamma \rangle$ whose supports are Easton sets i.e. sets of ordinals which are bounded in every regular cardinal. Let $\lambda_\gamma = \sup_{\beta<\gamma} \lambda_\beta$.
- For successor steps let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ where $\dot{\mathbb{Q}}_\alpha$ is the name for $1_{\mathbb{P}_\alpha}$ unless both:
  1. $\forall \beta < \alpha, \lambda_\beta < \alpha$
  2. $f(\alpha) = \langle \mathbb{Q}, \lambda \rangle$ where λ is an ordinal, $\mathbb{Q}$ a term in the forcing language of $\mathbb{P}_\alpha$ such that

$$\Vdash_{\mathbb{P}_\alpha} (\mathbb{Q} \text{ is a } \kappa\text{-directed closed p.o.})$$

  If so then set $\mathbb{Q}_\alpha = \mathbb{Q}$ and $\lambda_\alpha = \lambda$.

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To this end let $\gamma \geq \kappa$. Let $\lambda > |\,\mathrm{TC}(\mathbb{Q})|$ be such that $\Vdash_{\mathbb{P}_\kappa * \dot{\mathbb{Q}}} (\lambda \geq 2^{\gamma^{<\kappa}})$.

Since f is a Laver function, there is in V a normal ultrafilter U_λ such that $(j_\lambda f)(\kappa) = \langle \mathbb{Q}, \lambda \rangle$.

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In M_λ we have that $j_\lambda \langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ is a sequence of posets of length $j_\lambda \kappa$ and begins with $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$.

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In fact for $\kappa + 1 \leq \delta < \lambda = \sup_{\beta < \kappa} \lambda_\beta$ we have that $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{1}_{\mathbb{P}_\delta}$.

Lifting Elementary Embeddings

The remainder of the iteration (from λ to $j_\lambda(\kappa)$) will be by definition $\geq \lambda$-directed closed in $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ (and so also in V). Call it $\mathbb{R}$.

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Note that if $p \Vdash \phi$ then $j_\lambda(p) \Vdash j_\lambda(\phi)$ as j_λ is an elementary embedding. To lift an elementary embedding it suffices to show, given a generic G over $\mathbb{P}_\kappa$, that $j_\lambda''(G)$ is in the generic over $j_\lambda(\mathbb{P}_\kappa)$.

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We have that $\lambda > |\mathrm{TC}(\mathbb{Q})|$ and that $j_\lambda(\mathbb{Q})$ is λ directed closed and so there will be some s in $j_\lambda(\mathbb{Q})$ such that $s \leq j_\lambda(p)$ for all $p \in G$ i.e. s is a lower bound for $j_\lambda''(G)$. We call such an s a *master condition*.

Silver's argument

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Note that this ultrafilter is indeed an ultrafilter, and that it is normal. To see that it is normal we use an equivalent formulation of normality, namely that $\forall F$ a choice function on $\mathcal{P}_\kappa(\gamma)$ there's a set in U upon which F is constant.

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Thus κ is supercompact in $V^{\mathbb{P}_\kappa * \dot{\mathbb{Q}}}$ as required. □

To see that the above claims hold note that we may take a sequence $s \leq r_1 \leq r_2 \leq \ldots$ of length $< \lambda$ in $\mathbb{R}$ such that the following hold:

- For all $X \in \mathcal{P}_\kappa\gamma$ there is some α where r_α decides whether $j_\lambda''(\gamma) \in j_\lambda(X)$ or not i.e. whether $X \in U$.
- If $r_\alpha \Vdash (j_\lambda''(\gamma) \in j_\lambda(X))$ and if F is a choice function on X then there are $\delta, \nu < \lambda$ such that:

$$r_\delta \Vdash \left(j_\lambda F(j_\lambda''(\gamma)) = j_\lambda \nu\right).$$

  And so for and larger ρ, r_ρ will also force this to hold, and so there will be a set in U upon which F is constant. Thus the ultrafilter is normal.